Section 6.5

Section 6.5
Nonhomogeneous Systems

Theorem. If on an interval J,

$$Z' = AZ + F$$

for some particular Z then

$$X' = AX + F$$

on J if and only if

$$X = U + Z$$

for some solution U to the related homogeneous equation

$$X' = AX$$

on J.

Theorem. Suppose that each of A and F is continuous on an interval J. Let Φ be a fundamental matrix for $X' = AX$ on J. A particular function Z such that

$$Z' = AZ + F$$

is given by

$$Z(t) = \Phi(t) \int (\Phi(t))^{-1} F(t) dt$$

Example. Suppose that

$$A = \begin{pmatrix} 4 & -3 \\ 2 & -1 \end{pmatrix} \text{ and } F(t) = \begin{pmatrix} t \\ e^t \end{pmatrix}.$$

The eigenvalues of A are 1 and 2 and corresponding eigenvectors are $\begin{pmatrix} 1 \\ 1 \end{pmatrix}$ and $\begin{pmatrix} 3 \\ 2 \end{pmatrix}$.

A fundamental matrix Φ for $X' = AX$ is given by

$$\Phi(t) = \begin{pmatrix} e^t & 3e^{2t} \\ e^t & 2e^{2t} \end{pmatrix}.$$

$$(\Phi(t))^{-1} = \begin{pmatrix} -2e^{-t} & 3e^{-t} \\ e^{-2t} & -e^{-2t} \end{pmatrix}$$

$$(\Phi(t))^{-1}F(t) = \begin{pmatrix} 3 - 2te^{-t} \\ te^{-2t} - e^{-t} \end{pmatrix}$$

$$\int (\Phi(t))^{-1}F(t)dt = \int \begin{pmatrix} 3 - 2te^{-t} \\ te^{-2t} - e^{-t} \end{pmatrix} dt$$

$$= \begin{pmatrix} 3t + 2e^{-t} + 2te^{-t} \\ -\frac{1}{4}(e^{-2t} + 2te^{-2t} - 4e^{-t}) \end{pmatrix}$$

$$\Phi(t) \int (\Phi(t))^{-1}F(t)dt = \begin{pmatrix} \frac{1}{2}t + 3e^{t} + 3te^{t} + \frac{5}{4} \\ 3te^{t} + \frac{3}{2} + t + 2e^{t} \end{pmatrix}$$

Thus

$$X' = AX + F$$

if and only if

$$X(t) = c_1 e^{t} \begin{pmatrix} 1 \\ 1 \end{pmatrix} + c_2 e^{2t} \begin{pmatrix} 3 \\ 2 \end{pmatrix} + \begin{pmatrix} \frac{1}{2}t + 3e^{t} + 3te^{t} + \frac{5}{4} \\ 3te^{t} + \frac{3}{2} + t + 2e^{t} \end{pmatrix}$$

Additional Examples. See Section 6.5 of the text.

Suggested Problems. Do the odd numbers for Section 6.5.