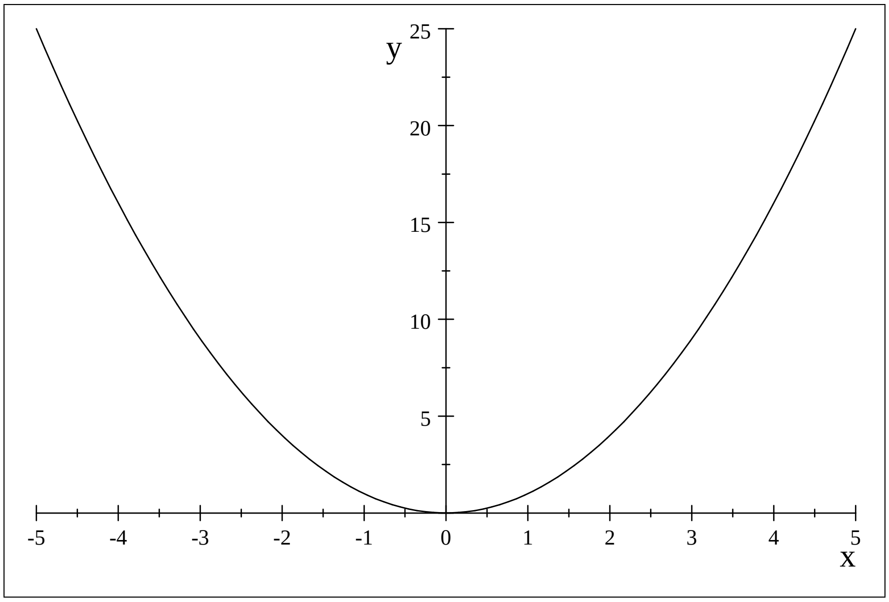
Even and Odd Functions

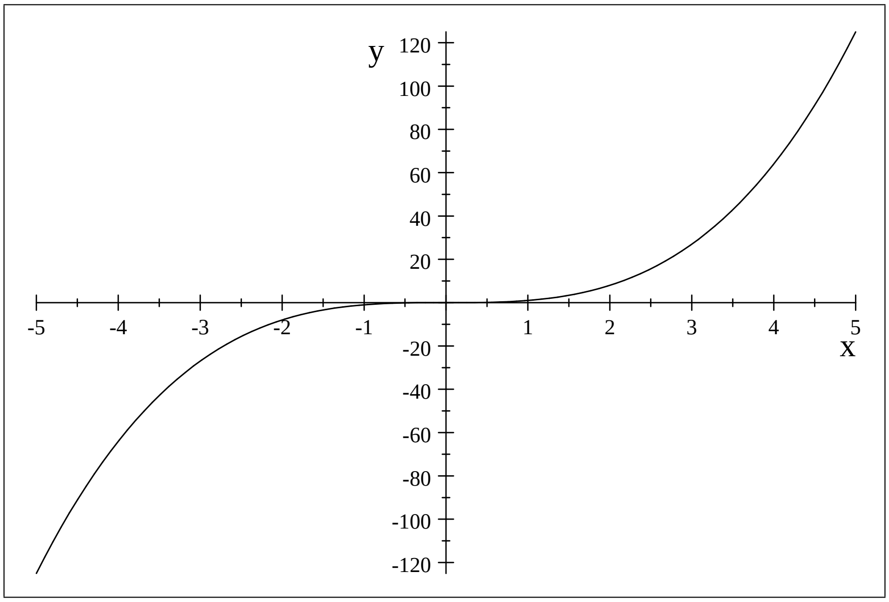
Definition. Saying that f is an even function means that $f(-x) = f(x)$ for all x in the domain of f. Saying that f is an odd function means that $f(-x) = -f(x)$ or $f(x) = -f(-x)$ for all x in the domain of f.

Note. The graph of an even function is symmetric about the y -axis.



An Even Function

The graph of an odd function is symmetric about the origin. (x, y) is on the graph if and only if $(-x, -y)$is on the graph.



An Odd Function

Note. If $f(x) = x^n$ then f is an even function when n is an even integer and f is an odd function when f is an odd integer. The cosine function is even and the sine function is odd.

Theorem. Suppose that each of f and g is an even function and each of u and v is an odd function all with the same domain D.

1. $f + g$ is an even function.
2. $u + v$ is an odd function (unlike with integers).
3. $f \cdot g$ is an even function.
4. $u \cdot v$ is an even function (unlike with integers).
5. $f \cdot u$ is an odd function (unlike with integers).

Proof of (5).

$$(f \cdot u)(-x) = f(-x)u(-x) = f(x) \cdot (-u(x)) = -f(x)u(x) = -(f \cdot u)(x)$$

for all x in D.

Suggested Problem. Prove Parts (1) - (4).

Note. Most functions are neither even nor odd. For example, if

$$f(x) = x + x^2$$

then

$$f(-1) = 0 \text{ while } f(1) = 2.$$

Of course, $0 \neq 2$ and $0 \neq -2$. So f is neither even nor odd.

However we do have the following fact.

Theorem. If the domain of f is symmetric about 0 (meaning x is in the domain if and only if $-x$ is in the domain) then f is the sum of an even function and an odd function.

Proof. Let

$$f_e(x) = \frac{1}{2}[f(x) + f(-x)] \text{ and } f_o(x) = \frac{1}{2}[f(x) - f(-x)]$$

Then
$$f_e(-x) = \frac{1}{2}[f(-x) + f(-(-x))] = \frac{1}{2}[f(x) + f(-x)] = f_e(x)$$
so f_e is even; and
$$f_o(-x) = \frac{1}{2}[f(-x) - f(-(-x))] = -\frac{1}{2}[f(x) - f(-x)] = -f_o(x)$$
so f_o is odd. Clearly
$$f(x) = f_e(x) + f_o(x)$$

Definition. f_e is called the even part of f and f_o is called the odd part of f

Theorem. If f is both even and odd, then f is the zero function on its domain.

Proof. $f(-x) = f(x)$ and $f(-x) = -f(x)$ so $f(x) = -f(x)$ for all x in the domain of f. Thus $2f(x) = 0$ implying $f(x) = 0$.

There is only one way to express a function as the sum of an even function and an odd function.

Theorem. Suppose that f is a function whose domain is symmetric about 0. If
$$f(x) = u_1(x) + u_2(x) = v_1(x) + v_2(x)$$
for all x in the domain of f, each of u_1 and v_1 is even, and each of u_2 and v_2 is odd then
$$u_1(x) = v_1(x) \text{ and } u_2(x) = v_2(x)$$
for all x in the domain of f.

Proof. If
$$f(x) = u_1(x) + u_2(x) = v_1(x) + v_2(x)$$
then
$$u_1(x) - v_1(x) = v_2(x) - u_2(x).$$
The left side of the last equation is even and the right side is odd so each side is both even and odd. This implies that each side is 0. Thus
$$u_1(x) = v_1(x) \text{ and } v_2(x) = u_2(x)$$

Note. From Calculus, we have

$$\int_b^a f(x)dx = -\int_a^b f(x)dx$$

and

$$\int_{h(a)}^{h(b)} f(x)dx = \int_a^b f(h(x))h'(x)dx.$$

Theorem. If f is an even function, then

$$\int_{-L}^{L} f(x)dx = 2\int_0^L f(x)dx.$$

Proof. Let $h(x) = -x$.Then

$$\begin{aligned}
\int_{-L}^{L} f(x)dx &= \int_{-L}^{0} f(x)dx + \int_0^L f(x)dx = \int_{h(L)}^{h(0)} f(x)dx + \int_0^L f(x)dx \\
&= \int_L^0 f(h(x))h'(x)dx + \int_0^L f(x)dx \\
&= \int_L^0 f(-x)(-1)dx + \int_0^L f(x)dx \\
&= -\int_L^0 f(x)dx + \int_0^L f(x)dx = \int_0^L f(x)dx + \int_0^L f(x)dx \\
&= 2\int_0^L f(x)dx
\end{aligned}$$

Theorem. If f is an odd function, then

$$\int_{-L}^{L} f(x)dx = 0.$$

Proof. Let $h(x) = -x$.

$$\begin{aligned}
\int_{-L}^{L} f(x)dx &= \int_{-L}^{0} f(x)dx + \int_0^L f(x)dx = \int_{h(L)}^{h(0)} f(x)dx + \int_0^L f(x)d \\
&= \int_L^0 f(h(x))h'(x)dx + \int_0^L f(x)dx \\
&= \int_L^0 f(-x)(-1)dx + \int_0^L f(x)dx \\
&= -\int_L^0 (-f(x))dx + \int_0^L f(x)dx \\
&= \int_L^0 f(x)dx + \int_0^L f(x)dx = -\int_0^L f(x)dx + \int_0^L f(x)dx \\
&= 0
\end{aligned}$$

Theorem. If f is an even function that is integrable over $[-L, L]$, the Fourier Series for f is $\{S_n\}$ where

$$S_n(x) = A_0 + \sum_{k=1}^{n} A_k \cos \frac{k\pi x}{L}$$

in which

$$A_0 = \frac{1}{L} \int_0^L f(x)dx$$

and

$$A_k = \frac{2}{L} \int_0^L f(x) \cos \frac{k\pi x}{L} dx$$

for $k = 1, 2, \ldots$.

Proof. According to the definition of a Fourier Series,

$$S_n(x) = A_0 + \sum_{k=1}^{n} \left[A_k \cos \frac{k\pi x}{L} + B_k \sin \frac{k\pi x}{L} \right]$$

where

$$A_0 = \frac{1}{2L} \int_{-L}^{L} f(x)dx,$$

$$A_k = \frac{1}{L} \int_{-L}^{L} f(x) \cos \frac{k\pi x}{L} dx \text{ for } k = 1, 2, \ldots, \text{ and}$$

$$B_k = \frac{1}{L} \int_{-L}^{L} f(x) \sin \frac{k\pi x}{L} dx \text{ for } k = 1, 2, \ldots.$$

Since the integrand is even

$$A_0 = \frac{1}{2L} \cdot 2 \int_0^L f(x)dx = \frac{1}{L} \int_0^L f(x)dx$$

and

$$A_k = \frac{1}{L} \int_{-L}^{L} f(x) \cos \frac{k\pi x}{L} dx = \frac{1}{L} \cdot 2 \int_0^L f(x) \cos \frac{k\pi x}{L} dx = \frac{2}{L} \int_0^L f(x) \cos \frac{k\pi x}{L} dx.$$

Since the integrand is odd (the product of an even function and an odd function is an odd function),

$$B_k = \frac{1}{L} \int_{-L}^{L} f(x) \sin \frac{k\pi x}{L} dx = 0$$

Example. If $f(x) = |x|$ for $-L \leq x \leq L$,since f is even, the Fourier Series for f is given by $\{S_n\}$ where

$$S_n(x) = A_0 + \sum_{k=1}^{n} A_k \cos \frac{k\pi x}{L}$$

in which

$$A_0 = \frac{1}{L}\int_0^L |x|dx = \frac{1}{L}\int_0^L xdx = \frac{L}{2}$$

and

$$\begin{aligned} A_k &= \frac{2}{L}\int_0^L f(x)\cos\frac{k\pi x}{L}dx = \frac{2}{L}\int_0^L x\cos\frac{k\pi x}{L}dx \\ &= \frac{2}{L}[[x\cdot\frac{L}{k\pi}\sin\frac{k\pi x}{L}]_{x=0}^{x=L} - \int_0^L 1\cdot\frac{L}{k\pi}\sin\frac{k\pi x}{L}dx] \\ &= \frac{2}{L}[0+\frac{L}{k\pi}\cdot\frac{L}{k\pi}[\cos\frac{k\pi x}{L}]_{x=0}^{x=L}] \\ &= \frac{2L}{k^2\pi^2}(\cos k\pi - \cos 0) \\ &= \frac{2L}{k^2\pi^2}((-1)^k - 1) \end{aligned}$$

for $k = 1, 2, \ldots$. So the Fourier Series is $\{S_n\}$ where

$$S_n(x) = \frac{L}{2} + \frac{2L}{\pi^2}\sum_{k=1}^n \frac{(-1)^k - 1}{k^2}\cos\frac{k\pi x}{L}.$$

Theorem. If f is an odd function that is integrable over $[-L, L]$, the Fourier Series for f is $\{S_n\}$ where

$$S_n(x) = \sum_{k=1}^n B_k \sin\frac{k\pi x}{L}$$

in which

$$B_k = \frac{2}{L}\int_0^L f(x)\sin\frac{k\pi x}{L}dx$$

for $k = 1, 2, \ldots$.

Proof. According to the definition of a Fourier Series,

$$S_n(x) = A_0 + \sum_{k=1}^n \left[A_k\cos\frac{k\pi x}{L} + B_k\sin\frac{k\pi x}{L}\right]$$

where

$$A_0 = \frac{1}{2L}\int_{-L}^L f(x)dx,$$

$$A_k = \frac{1}{L}\int_{-L}^L f(x)\cos\frac{k\pi x}{L}dx \text{ for } k = 1, 2, \ldots, \text{ and}$$

$$B_k = \frac{1}{L}\int_{-L}^L f(x)\sin\frac{k\pi x}{L}dx \text{ for } k = 1, 2, \ldots.$$

Since the integrand is odd,

$$A_0 = 0$$

and

$$A_k = 0$$

for $k = 1, 2, \ldots$.Since the integrand is even,

$$\begin{aligned}
B_k &= \frac{1}{L}\int_{-L}^{L} f(x)\sin\frac{k\pi x}{L}dx \\
&= \frac{2}{L}\int_{0}^{L} f(x)\sin\frac{k\pi x}{L}dx = \frac{2}{L}\int_{0}^{L} x\sin\frac{k\pi x}{L}dx \\
&= \frac{2}{L}[[x\cdot(-\frac{L}{k\pi})\cos\frac{k\pi x}{L}]_{x=0}^{x=L} + \int_0^L 1\cdot\frac{L}{k\pi}\cos\frac{k\pi x}{L}dx] \\
&= \frac{2}{L}[\frac{-L^2}{k\pi}(-1)^k + \frac{L^2}{k^2\pi^2}[\sin\frac{k\pi x}{L}]_{x=0}^{x=L}] \\
& \frac{2L}{k\pi}(-1)^{k+1}
\end{aligned}$$

So the Fourier Series is $\{S_n\}$ where

$$S_n(x) = \frac{2L}{\pi}\sum_{k=1}^{n}\frac{(-1)^k}{k}\sin\frac{k\pi x}{L}.$$