Fourier Series when the Limit Function is Continuous

<u>Theorem</u>: If a sequence of continuous functions converges uniformly, the limit function is continuous.

<u>Note</u>: Each S_n in a Fourier Series is continuous because it is a linear combination of sines and cosines.

<u>Theorem</u>: Suppose that f is piecewise smooth on $[-L, L]$. If f is discontinuous and/or $f(-L) \neq f(L)$, the Fourier series for f does not converge uniformly

<u>Theorem</u>: If f is continuous and piecewise smooth on $[-L, L]$, and $f(-L) = f(L)$, the Fourier Series for f converges uniformly.

<u>Theorem</u>: Suppose that the Fourier Series for a function defined on $[-L, L]$ is $\{S_n\}$ where

$$S_n(x) = A_0 + \sum_{k=1}^{\infty}\left[A_k \cos\frac{k\pi x}{L} + B_k \sin\frac{k\pi x}{L}\right]$$

and suppose that each of $\sum_{k=0}^{\infty} |A_k|$ and $\sum_{k=1}^{\infty} |B_k|$ exist and are finite. It follows that $\{S_n\}$ converges uniformly on $\mathbb{R}$.

Proof: This follows from the Weierstrass M test because

$$\left|A_k \cos\frac{k\pi x}{L} + B_k \sin\frac{k\pi x}{L}\right|$$

$$\leq \left|A_k \cos\frac{k\pi x}{L}\right| + \left|B_k \sin\frac{k\pi x}{L}\right|$$

$$\leq |A_k| \cdot \left|\cos\frac{k\pi x}{L}\right| + |B_k| \cdot \left|\sin\frac{k\pi x}{L}\right|$$

$$\leq |A_k| + |B_k|$$