Laplace's Equation in a Rectangle

Example 1

Problem: Find the solution u to Laplace's equation in the rectangle $[0,\pi]\times[0,\pi]$ such that $u(x,y)=F(x,y)$on the boundary of $[0,\pi]\times[0,\pi]$ when $F(x,y)=x^2$. Stating the problem in more detail, it becomes that of finding u such that

$$\frac{\partial^2 u}{\partial x^2}(x,y)+\frac{\partial^2 u}{\partial y^2}(x,y)=0 \text{ for } (x,y) \text{ in } [0,\pi]\times[0,\pi],$$

$$u(0,y)=0 \text{ when } 0\le y\le\pi,$$
$$u(\pi,y)=\pi^2 \text{ when } 0\le y\le\pi,$$
$$u(x,0)=x^2 \text{ when } 0\le x\le\pi,$$

and

$$u(x,\pi)=x^2 \text{ when } 0\le x\le\pi.$$

Solution. First find the function v such that

$$v(x,y)=ax+by+cxy+d$$

and $v(x,y)=F(x,y)$ at each of the four corners of $[0,\pi]\times[0,\pi]$. Thus v is to satisfy

$$\begin{aligned} v(0,0) &= 0\\ v(\pi,0) &= \pi^2\\ v(0,\pi) &= 0\\ v(\pi,\pi) &= \pi^2 \end{aligned}$$

so

$$\begin{aligned} d &= 0\\ a\pi &= \pi^2 \text{ implying } a=\pi\\ b\pi &= 0 \text{ amplifying } b=0\\ \pi^2+c\pi^2 &= \pi^2 \text{ implying } c=0. \end{aligned}$$

Thus

$$v(x,y)=\pi x$$

Next let

$$w(x,y)=u(x,y)-v(x,y)$$

and state the problem for w. Using the definition of v and the fact that v is a solution to Laplace's equation we have

$$\frac{\partial^2 w}{\partial x^2}(x,y) + \frac{\partial^2 w}{\partial y^2}(x,y) = 0 \text{ for } (x,y) \text{ in } [0,\pi] \times [0,\pi],$$

$$\begin{aligned}
w(0,y) &= u(0,y) - v(0,y) = 0 - 0 = 0, \\
w(\pi,y) &= u(\pi,y) - v(\pi,y) = \pi^2 - \pi^2 = 0, \\
w(x,0) &= u(x,0) - v(x,0) = x^2 - \pi x, \text{ and} \\
w(x,\pi) &= u(x,\pi) - v(x,\pi) = x^2 - \pi x.
\end{aligned}$$

To find w we follow the procedure given in Laplace Equation Problem I to find w_1 ($w_1$is the same as w^{lower}) such that

$$\begin{aligned}
\frac{\partial^2 w_1}{\partial x^2}(x,y) + \frac{\partial^2 w_1}{\partial y^2}(x.y) &= 0 \text{ for } 0 \le x \le \pi \text{ and } 0 \le y \le \pi, \\
w_1(0,y) &= 0 \text{ for } 0 \le y \le \pi, \\
w_1(\pi,y) &= 0 \text{ for } 0 \le y \le \pi, \\
w_1(x,\pi) &= 0 \text{ for } 0 \le x \le \pi, \text{ and} \\
w_1(x,0) &= x^2 - \pi x \text{ for } 0 \le x \le \pi,
\end{aligned}$$

the procedure given in Laplace Equation Problem II to find w_2 (w_2 is the same as w^{upper}) such that

$$\begin{aligned}
\frac{\partial^2 w_2}{\partial x^2}(x,y) + \frac{\partial^2 w_2}{\partial y^2}(x.y) &= 0 \text{ for } 0 \le x \le \pi \text{ and } 0 \le y \le \pi, \\
w_2(0,y) &= 0 \text{ for } 0 \le y \le \pi, \\
w_2(\pi,y) &= 0 \text{ for } 0 \le y \le \pi, \\
w_2(x,\pi) &= x^2 - \pi x \text{ for } 0 \le x \le \pi, \text{ and} \\
w_2(x,0) &= 0 \text{ for } 0 \le x \le \pi,
\end{aligned}$$

the procedure given in Laplace Equation Problem III to find w_3 (w_3 is the same as w^{left}) such that

$$\begin{aligned}
\frac{\partial^2 w_3}{\partial x^2}(x,y) + \frac{\partial^2 w_3}{\partial y^2}(x.y) &= 0 \text{ for } 0 \le x \le \pi \text{ and } 0 \le y \le \pi, \\
w_3(0,y) &= 0 \text{ for } 0 \le y \le \pi, \\
w_3(\pi,y) &= 0 \text{ for } 0 \le y \le \pi, \\
w_3(x,\pi) &= 0 \text{ for } 0 \le x \le \pi, \text{ and} \\
w_3(x,0) &= 0 \text{ for } 0 \le x \le \pi,
\end{aligned}$$

and the procedure given in Laplace Equation Problem IV to find w_4 ($w_4$is the same as w^{right})

such that

$$\begin{aligned}
\frac{\partial^2 w_4}{\partial x^2}(x,y) + \frac{\partial^2 w_4}{\partial y^2}(x.y) &= 0 \text{ for } 0 \le x \le \pi \text{ and } 0 \le y \le \pi, \\
w_4(0,y) &= 0 \text{ for } 0 \le y \le \pi, \\
w_4(\pi,y) &= 0 \text{ for } 0 \le y \le \pi, \\
w_4(x,\pi) &= 0 \text{ for } 0 \le x \le \pi, \text{ and} \\
w_4(x,0) &= 0 \text{ for } 0 \le x \le \pi.
\end{aligned}$$

Then let

$$w = w_1 + w_2 + w_3 + w_4.$$

Using the formulas that have been worked out for each case and using the fact (integrate by parts) that

$$\frac{2}{\pi \sinh k\pi} \int_0^\pi (x^2 - \pi x) \sin kx dx = \frac{4((-1)^k - 1)}{k^3 \pi \sinh k\pi}$$

we find that

$$w_1(x,y) = \sum_{k=1}^{\infty} \frac{4((-1)^k - 1)}{k^3 \pi \sinh k\pi} \sin kx \sinh k(\pi - y),$$

$$w_2(x,y) = \sum_{k=1}^{\infty} \frac{4((-1)^k - 1)}{k^3 \pi \sinh k\pi} \sin kx \sinh ky,$$

and

$$w_3(x,y) = w_4(x,y) = 0$$

for all (x,y) in $[0,\pi] \times [0,\pi]$.Thus since $u = v + w$,

$$u(x,y) = \pi x + \frac{4}{\pi} \sum_{k=1}^{\infty} \frac{((-1)^k - 1)}{k^3 \sinh k\pi} (\sin kx)(\sinh k(\pi - y) + \sinh ky)$$

for all (x,y) in $[0,\pi] \times [0,\pi]$.