Linear Algebra

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1 Section 5.6

1.1 The Inverse of a Matrix

Definition 1 *Saying that an $n \times n$ matrix A is invertible or has an inverse means that there there is an $n \times n$ matrix B such that*

$$AB = I_n$$

where I_n is the $n \times n$ identity matrix.

Remark 2 *Not every matrix is invertible.*

$$\begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} a+c & b+d \\ 0 & 0 \end{pmatrix} \neq \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

so the matrix

$$\begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix}$$

is not invertible.

Theorem 3 *If the $n \times n$ matrix A is invertible, there is only one matrix B such that*

$$AB = I_n.$$

Remark 4 *The matrix B must also be $n \times n$.*

Definition 5 *If the $n \times n$ matrix A is invertible, the matrix B such that $AB = I_n$ is called the matrix inverse to A or the inverse of A or A-inverse and is denoted by A^{-1}.*

Theorem 6 *If each of A and B is an $n \times n$ matrix and $BA = I_n$ then A is invertible and $B = A^{-1}$.*

Theorem 7 *If A is a 2×2 matrix with*

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

then A is invertible if and only if $\det A = ad - bc \neq 0$*, in which case,*

$$A^{-1} = \frac{1}{ad - bc} \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}.$$

You should memorize this formula.

Remark 8 *Suppose that A is an $n \times n$ matrix. Here is a procedure to determine whether or not A is invertible and find A^{-1} if it exists. Form the $n \times 2n$ matrix*

$$\begin{bmatrix} A & I_n \end{bmatrix}$$

and perform elementary row operations to put it into reduced row-echelon form. If the result is

$$\begin{bmatrix} I_n & B \end{bmatrix}$$

then A is invertible and $B = A^{-1}$. If the result is

$$\begin{bmatrix} C & D \end{bmatrix}$$

where each of C and D is an $n \times n$ matrix and $C \neq I_n$ then A is not invertible.

Example 9 *Let*

$$A = \begin{pmatrix} 1 & 0 & 2 \\ 2 & -1 & 3 \\ 4 & 1 & 8 \end{pmatrix}.$$

Then

$$\begin{bmatrix} A & I_3 \end{bmatrix} = \begin{pmatrix} 1 & 0 & 2 & 1 & 0 & 0 \\ 2 & -1 & 3 & 0 & 1 & 0 \\ 4 & 1 & 8 & 0 & 0 & 1 \end{pmatrix}$$

$$\overrightarrow{-2R_1 + R_2 \to R_2 \; and \; -4R_1 + R_3 \to R_3}$$

$$\begin{pmatrix} 1 & 0 & 2 & 1 & 0 & 0 \\ 0 & -1 & -1 & -2 & 1 & 0 \\ 0 & 1 & 0 & -4 & 0 & 1 \end{pmatrix}$$

$$\overrightarrow{R_2 \leftrightarrow R_3}$$

$$\begin{pmatrix} 1 & 0 & 2 & 1 & 0 & 0 \\ 0 & 1 & 0 & -4 & 0 & 1 \\ 0 & -1 & -1 & -2 & 1 & 0 \end{pmatrix}$$

$$\overrightarrow{R_2 + R_3 \to R_3}$$

$$\begin{pmatrix} 1 & 0 & 2 & 1 & 0 & 0 \\ 0 & 1 & 0 & -4 & 0 & 1 \\ 0 & 0 & -1 & -6 & 1 & 1 \end{pmatrix}$$

$$\overrightarrow{-R_3 \to R_3}$$

$$\begin{pmatrix} 1 & 0 & 2 & 1 & 0 & 0 \\ 0 & 1 & 0 & -4 & 0 & 1 \\ 0 & 0 & 1 & 6 & -1 & -1 \end{pmatrix}$$

$$\overrightarrow{-2R_3 + R_1 \rightarrow R_1}$$

$$\begin{pmatrix} 1 & 0 & 0 & -11 & 2 & 2 \\ 0 & 1 & 0 & -4 & 0 & 1 \\ 0 & 0 & 1 & 6 & -1 & -1 \end{pmatrix}.$$

Since the left half of this matrix is I_3, The matrix A is invertible and $A^{-1}$is the right half.

$$A^{-1} = \begin{pmatrix} -11 & 2 & 2 \\ -4 & 0 & 1 \\ 6 & -1 & -1 \end{pmatrix}.$$

Example 10 *Let*

$$A = \begin{pmatrix} 1 & 3 & -4 \\ 1 & 5 & -1 \\ 3 & 13 & -6 \end{pmatrix}.$$

Then

$$\begin{bmatrix} A & I_3 \end{bmatrix} = \begin{pmatrix} 1 & 3 & -4 & 1 & 0 & 0 \\ 1 & 5 & -1 & 0 & 1 & 0 \\ 3 & 13 & -6 & 0 & 0 & 1 \end{pmatrix}$$

$$\overrightarrow{-R_1 + R_2 \rightarrow R_2 \text{ and } -3R_1 + R_3 \rightarrow R_3}$$

$$\begin{pmatrix} 1 & 3 & -4 & 1 & 0 & 0 \\ 0 & 2 & 3 & -1 & 1 & 0 \\ 0 & 4 & 6 & -3 & 0 & 1 \end{pmatrix}$$

$$\overrightarrow{\frac{1}{2}R_2 \to R_2}$$

$$\begin{pmatrix} 1 & 3 & -4 & 1 & 0 & 0 \\ 0 & 1 & 3/2 & -1/2 & 1/2 & 0 \\ 0 & 4 & 6 & -3 & 0 & 1 \end{pmatrix}$$

$$\overrightarrow{-4R_2 + R_3 \to R_3}$$

$$\begin{pmatrix} 1 & 3 & -4 & 1 & 0 & 0 \\ 0 & 1 & 3/2 & -1/2 & 1/2 & 0 \\ 0 & 0 & 0 & -1 & -2 & 1 \end{pmatrix}$$

$$\overrightarrow{-R_3 \to R_3}$$

$$\begin{pmatrix} 1 & 3 & -4 & 1 & 0 & 0 \\ 0 & 1 & 3/2 & -1/2 & 1/2 & 0 \\ 0 & 0 & 0 & 1 & 2 & -1 \end{pmatrix}$$

$$\overrightarrow{-3R_2 + R_1 \to R_1}\ ,\overrightarrow{\frac{1}{2}R_3 + R_2 \to R_2}\ \textit{and}\ \overrightarrow{-R_3 + R_1 \to R_1}$$

$$\begin{pmatrix} 1 & 0 & -17/2 & 0 & -13/2 & 5/2 \\ 0 & 1 & 3/2 & 0 & 3/2 & -1/2 \\ 0 & 0 & 0 & 1 & 2 & -1 \end{pmatrix}.$$

The left half of this matrix is not I_3 so the matrix A is not invertible

Remark 11 *If A is $n \times n$ and invertible the system*

$$AX = B$$

which is equivalent to (1) in Section 5.3 in the notes can be solved by multiplying each side of the equation on the left by A^{-1}.

$$A^{-1}AX = A^{-1}B$$

$$I_n X = A^{-1}B$$
$$X = A^{-1}B$$

Example 12 *Use this method to solve the system*

$$\begin{aligned} x + 2y - z &= 2 \\ x + y + 2z &= 0 \\ x - y - z &= 1 \end{aligned}$$

Solution. *The matrix formulation of this system is*

$$AX = B$$

where

$$A = \begin{pmatrix} 1 & 2 & -1 \\ 1 & 1 & 2 \\ 1 & -1 & -1 \end{pmatrix}, \ X = \begin{bmatrix} x \\ y \\ z \end{bmatrix}, \ \text{and } B = \begin{bmatrix} 2 \\ 0 \\ 1 \end{bmatrix}$$

The reduced row echelon form of

$$\begin{bmatrix} A & I_3 \end{bmatrix} = \begin{pmatrix} 1 & 2 & -1 & 1 & 0 & 0 \\ 1 & 1 & 2 & 0 & 1 & 0 \\ 1 & -1 & -1 & 0 & 0 & 1 \end{pmatrix}$$

is

$$\begin{pmatrix} 1 & 0 & 0 & \frac{1}{9} & \frac{1}{3} & \frac{5}{9} \\ 0 & 1 & 0 & \frac{1}{3} & 0 & -\frac{1}{3} \\ 0 & 0 & 1 & -\frac{2}{9} & \frac{1}{3} & -\frac{1}{9} \end{pmatrix}$$

so

$$A^{-1} = \begin{pmatrix} 1/9 & 1/3 & 5/9 \\ 1/3 & 0 & -1/3 \\ -2/9 & 1/3 & -1/9 \end{pmatrix}.$$

Thus

$$X = A^{-1}B = \begin{pmatrix} 1/9 & 1/3 & 5/9 \\ 1/3 & 0 & -1/3 \\ -2/9 & 1/3 & -1/9 \end{pmatrix} \begin{bmatrix} 2 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} \frac{7}{9} \\ \frac{1}{3} \\ -\frac{5}{9} \end{bmatrix}.$$

Thus

$$x = 7/9,\ y = 1/3,\ \textit{and}\ z = -5/9$$

1.2 Determinants

Definition 13 *If A is a 1×1 matrix with $A = (a)$, the determinant of A is a.If A is a 2×2 matrix with*

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix},$$

the determinant of A is $ad - bc$.

Definition 14 *The determinant of a matrix A is denoted* $\det A$.

Definition 15 *When A is an $n \times n$ matrix with $n \geq 2$,and each of i and j is a positive integer with $i \leq n$ and $j \leq n$, then $A(i \mid j)$ is the $(n-1) \times (n-1)$ matrix obtained by removing the ith row and jth column from A. (You will not find this notation in the test.)*

We define the determinant of an $n \times n$ matrix recursively by expansion across the top row.

Definition 16 *When A is an $n \times n$ matrix with $n > 2$*

$$\det A = \sum_{j=1}^{n} (-1)^{1+j} A_{1j} \det A(1 \mid j).$$

Example 17

$$\det \begin{pmatrix} a & b & c \\ d & e & f \\ g & h & i \end{pmatrix}$$

$$= a \det \begin{pmatrix} e & f \\ h & i \end{pmatrix} - b \det \begin{pmatrix} d & f \\ g & i \end{pmatrix} + c \det \begin{pmatrix} d & e \\ g & h \end{pmatrix}$$

It is also true that one can expand across any row or down any column.

Theorem 18 *When A is an $n \times n$ matrix with $n > 2$ and i is an integer with $1 \leq i \leq n$ then*

$$\det A = \sum_{j=1}^{n} (-1)^{i+j} A_{ij} \det A(i \mid j). \tag{1}$$

When j is an integer with $1 \leq j \leq n$ then

$$\det A = \sum_{i=1}^{n} (-1)^{i+j} A_{ij} \det A(i \mid j).$$

Example 19 *Expanding across the second row we have*

$$\det \begin{pmatrix} a & b & c \\ d & e & f \\ g & h & i \end{pmatrix}$$

$$= -d \det \begin{pmatrix} b & c \\ h & i \end{pmatrix} + e \det \begin{pmatrix} a & c \\ g & i \end{pmatrix} - f \det \begin{pmatrix} a & b \\ g & h \end{pmatrix}$$

Expanding down the third column we have

$$\det \begin{pmatrix} a & b & c \\ d & e & f \\ g & h & i \end{pmatrix}$$

$$= c \det \begin{pmatrix} d & e \\ g & h \end{pmatrix} - f \det \begin{pmatrix} a & b \\ g & h \end{pmatrix} + i \det \begin{pmatrix} a & b \\ d & e \end{pmatrix}$$

Remark 20 *When using this procedure, you should pick the row or column with the largest number of zeros.*

A more efficient way to find the determinant of a large matrix is to use elementary row operations to transform the matrix into triangular form. First some definitions.

Definition 21 *The main diagonal entries of an $n \times n$ matrix A are the entries $A_{11}, A_{22}, \ldots, A_{nn}$.*

Definition 22 *Saying that a matrix is upper triangular means that all entries below the main diagonal are zero. Saying that a matrix is lower triangular means that all entries above the main diagonal are zero. Saying that a matrix is triangular means that it is upper triangular or that it is lower triangular.*

Example 23 *Continuing to expand down the first column at each stage, we have*

$$\det\begin{pmatrix} a & b & c & d \\ 0 & e & f & g \\ 0 & 0 & h & i \\ 0 & 0 & 0 & j \end{pmatrix} = a\det\begin{pmatrix} e & f & g \\ 0 & h & i \\ 0 & 0 & j \end{pmatrix} = ae\begin{pmatrix} h & i \\ 0 & j \end{pmatrix} = aehj.$$

Example 24 *Continuing to expand across the first row at each stage, we have*

$$\det\begin{pmatrix} a & 0 & 0 & 0 \\ b & c & 0 & 0 \\ d & e & f & 0 \\ g & h & i & j \end{pmatrix} = a\det\begin{pmatrix} c & 0 & 0 \\ e & f & 0 \\ h & i & j \end{pmatrix} = ac\det\begin{pmatrix} f & 0 \\ i & j \end{pmatrix} = acfj$$

These examples illustrate the following theorem.

Theorem 25 *The determinant of a triangular matrix is the product of its diagonal entries.*

We need to know how elementary row operations affect the determinant of a matrix.

Theorem 26 *Suppose that A is an $n \times n$ matrix and B comes from A by an elementary row operation.*

1. *If $A \to B$ by $R_i \leftrightarrow R_j$ where $i \neq j$ then $\det B = -\det A$.*
2. *If $A \to B$ by $cRi \to Ri$ where $c \neq 0$ then $\det B = c \det A$.*
3. *If $A \to B$ by $cRi + R_j \to R_j$ then $\det B = \det A$.*

Example 27 *Find $\det A$ using elementary row operations when*

$$A = \begin{pmatrix} 6 & 1 & 12 \\ 1 & -4 & 3 \\ 4 & 1 & 8 \end{pmatrix}$$

Solution. *Let $A_1 = A$*

$$\overrightarrow{R_1 \leftrightarrow R_2}$$

$$\begin{pmatrix} 1 & -4 & 3 \\ 6 & 1 & 12 \\ 4 & 1 & 8 \end{pmatrix} = A_2$$

$$\overrightarrow{-6R_1 + R_2 \to R_2}\ \overrightarrow{-4R_1 + R_3 \to R_3}$$

$$\begin{pmatrix} 1 & -4 & 3 \\ 0 & 25 & -6 \\ 0 & 17 & -4 \end{pmatrix} = A_3$$

$$\overrightarrow{-\frac{17}{25}R_2 + R_3 \to R_3}$$

$$\begin{pmatrix} 1 & -4 & 3 \\ 0 & 25 & -6 \\ 0 & 0 & \frac{2}{25} \end{pmatrix} = A_4$$

$$\det A_2 = -\det A_1$$
$$\det A_3 = \det A_2 = -\det A_1$$
$$\det A_4 = \det A_3 = -\det A_1$$
$$\det A_4 = (1)(25)(\frac{2}{25}) = 2$$

Thus

$$\det A = \det A_1 = -\det A_4 = -2.$$

Definition 28 *Saying that an $n \times n$ matrix is singular means that its determinant is zero. Saying that it is nonsingular means that its determinant is nonzero.*

Theorem 29 *An $n \times n$ matrix is invertible if and only if it is nonsingular.*

1.3 Additional Properties of Determinants

Definition 30 *When A is $m \times n$, the transpose of A denoted A^T is the $n \times m$ matrix whose $i-j$ entry is A_{ji} for $i = 1, \ldots, n$ and $j = 1, \ldots, m$.*

Example 31

$$\begin{pmatrix} a & b & c & d \\ e & f & g & h \\ i & j & k & l \end{pmatrix}^T = \begin{pmatrix} a & e & i \\ b & f & j \\ c & g & k \\ d & h & l \end{pmatrix}.$$

Example 32

$$\begin{pmatrix} a & b & c \end{pmatrix}^T = \begin{pmatrix} a \\ b \\ c \end{pmatrix}.$$

Theorem 33 *When A is $n \times n$,*

$$\det A^T = \det A.$$

Theorem 34 *If the $n \times n$ matrix A has an all zero row or an all zero column, then* $\det A = 0$.

Theorem 35 *If the $n \times n$ matrix A has two identical rows or two identical colums, then* $\det A = 0$.

Theorem 36 *If each of A and B is $n \times n$,then*

$$\det(AB) = \det A \det B.$$

1.4 Cramer's Rule

Theorem 37 *Suppose that the $n \times n$ matrix A is nonsingular. The system*

$$\begin{array}{ccccccccc} A_{11}x_1 & + & A_{12}x_2 & + & \cdots & + & A_{1n}x_n & = & B_1 \\ A_{21}x_1 & + & A_{22}x_2 & + & \cdots & + & A_{2n}x_n & = & B_2 \\ \vdots & & \vdots & & & & \vdots & & \vdots \\ A_{n1}x_1 & + & A_{n2}x_2 & + & \cdots & + & A_{nn}x_n & = & Bn \end{array}$$

has a unique solution $(x_1, x_2, \ldots, x_n)$ where

$$x_j = \frac{\det A_j}{\det A} \text{ for } j = 1, \ldots, n$$

where A_j is the $n \times n$ matrix obtained by replacing the jth column of A with

$$\begin{bmatrix} B_1 \\ B_2 \\ \vdots \\ Bn \end{bmatrix}.$$

Example 38 *Use Cramer's Rule to solve the system.*

$$\begin{aligned} x + 2y - z &= 2 \\ x + y + 2z &= 0 \\ x - y - z &= 1 \end{aligned}$$

Solution.

$$\det \begin{pmatrix} 1 & 2 & -1 \\ 1 & 1 & 2 \\ 1 & -1 & -1 \end{pmatrix} = 9$$

and

$$\det \begin{pmatrix} 2 & 2 & -1 \\ 0 & 1 & 2 \\ 1 & -1 & -1 \end{pmatrix} = 7$$

so

$$x = 7/9.$$

$$\det \begin{pmatrix} 1 & 2 & -1 \\ 1 & 0 & 2 \\ 1 & 1 & -1 \end{pmatrix} = 3$$

so

$$y = 3/9 = 1/3.$$

$$\det \begin{pmatrix} 1 & 2 & 2 \\ 1 & 1 & 0 \\ 1 & -1 & 1 \end{pmatrix} = -5$$

so

$$z = -5/9$$

1.5 Some Connections

Theorem 39 *Suppose that A is an $n \times n$ matrix. Each two of the following statements are equivalent.*

1. The system $AX = B$ has a unique solution for each n-dimensional column matrix B.
2. A is invertible.
3. The reduced row-echelon form of A is I_n.
4. $\det A \neq 0$
5. The rank of A is n.

Theorem 40 *If the $n \times n$ matrix A is invertible, then the i-th row and j-th column entry of A^{-1} is*

$$\frac{1}{\det A}(-1)^{i+j} \det A(j|i)$$

for $i = 1, \ldots, n$ and $j = 1, \ldots, n$.

1.6 Suggested Problems

Do the odd numbered problems for Section 5.6