The Wave Equation for a Vibrating Rectangular Membrane

We consider the wave equation for a vibrating rectangular membrane whose edges are fixed to a flat frame. The problem is as follows.

$$\begin{aligned}
\frac{\partial^2 u}{\partial t^2}(x,y,t) &= c^2\left(\frac{\partial^2 u}{\partial x^2}(x,y,t) + \frac{\partial^2 u}{\partial y^2}(x,y,t)\right) \qquad (1)\\
\text{for } 0 &\leq x \leq L,\ 0 \leq y \leq H,\ \text{and all } t,\\
u(x,y,t) &= 0 \text{ for } (x,y) \text{ on the boundary of } [0,L]\times[0,H] \text{ and all } t, \qquad (2)\\
u(x,y,0) &= \alpha(x,y) \text{ for } 0 \leq x \leq L,\ \text{and } 0 \leq y \leq H,\ \text{and} \qquad (3)\\
\frac{\partial u}{\partial t}(x,y,0) &= \beta(x,y) \text{ for } 0 \leq x \leq L,\ \text{and } 0 \leq y \leq H. \qquad (4)
\end{aligned}$$

Solution. Suppose that

$$u(x,t) = \varphi(x,y)h(t).$$

From (1) it follows that

$$\begin{aligned}
\varphi(x,y)h''(t) &= c^2\left(\frac{\partial^2 \varphi}{\partial x^2}(x,y) + \frac{\partial^2 \varphi}{\partial y^2}(x,y)\right)h(t) \qquad (5)\\
\text{for } 0 &\leq x \leq L,\ 0 \leq x \leq H \text{ and all } t \text{ in } \mathbb{R}.
\end{aligned}$$

Assuming for now that $\varphi(x,y)h(t) \neq 0$ and dividing each side of (5) by c^2 times this quantity, we have

$$\frac{\nabla^2\varphi(x,y)}{\varphi(x,y)} = \frac{1}{c^2}\frac{h''(t)}{h(t)} \text{ for } 0 \leq x \leq L,\ 0 \leq x \leq H \text{ and all } t \text{ in } \mathbb{R}. \qquad (6)$$

Letting $-\lambda$ be the common constant value for the left and right sides of (6), it follows that

$$-\nabla^2\varphi(x,y) = \lambda\varphi(x,y) \text{ for } 0 \leq x \leq L \text{ and } 0 \leq y \leq H \qquad (7)$$

and

$$h''(t) + c^2\lambda h(t) = 0 \text{ for all } t \text{ in } \mathbb{R}. \qquad (8)$$

It should be noted that if (7) and (8) hold and $u(x,y,t) = \varphi(x,y)h(t)$ then (1) will hold and the assumption that $\varphi(x,y)h(t) \neq 0$ is not needed. If u is not the zero function ((3) and (4) will not hold if it is), it follows from (2) that

$$\varphi(x,y) = 0 \text{ for all } (x,y) \text{ on the boundary of the rectangle } [0,L]\times[0,H] \qquad (9)$$

A proper listing of eigenvalues and eigenfuncions (See 'A Two-dimensional Rectangular Eigenvalue Problem') for (7) and (9) is given by

$$\{\lambda_{kj}\}_{k=1,j=1}^{\infty} \text{ and } \{\varphi_{kj}\}_{k=1,j=1}^{\infty}$$

where

$$\lambda_{kj} = (\frac{k\pi}{L})^2 + (\frac{j\pi}{H})^2$$

and

$$\varphi_{kj}(x,y) = \sin\frac{k\pi x}{L}\sin\frac{j\pi y}{H}$$

for $0 \le x \le L$, $0 \le y \le H$, $k = 1, 2, \ldots$, and $j = 1, 2, \ldots$.

When $\lambda = \lambda_{kj}$ the solutions to (8) are linear combinations of h_{1kj} and h_{2kj} where

$h_{1kj}(t) = \cos\sqrt{\lambda_{kj}}ct,$ and $h_{2kj}(t) = \sin\sqrt{\lambda_{kj}}ct$ for $k = 1, 2, \ldots$ and $j = 1, 2, \ldots$.

Considering the possible combinations, we expect the solution to (1)-(4) to be given by

$$u(x,y,t) = \sum_{k=1}^{\infty}\sum_{j=1}^{\infty}\varphi_{kj}(x,y)\left(A_{kj}h_{1kj}(t) + B_{kj}h_{2kj}(t)\right)$$

and

$$\frac{\partial u}{\partial t}(x,y,t) = \sum_{k=1}^{\infty}\sum_{j=1}^{\infty}\varphi_{kj}(x,y)\left(A_{kj}h'_{1kj}(t) + B_{kj}h'_{2kj}(t)\right)$$

Note that $h_{1kj}(0) = 1$, $h_{2kj}(0) = 0$, $h'_{1kj}(0) = 0$, and $h'_{2kj}(0) = \sqrt{\lambda_{kj}}c$ for for $k = 1, 2, \ldots$and $j = 1, 2, \ldots$.

Condition (3) will hold if and only if

$$\alpha = \sum_{k=1}^{\infty}\sum_{j=1}^{\infty}A_{kj}\varphi_{kj}$$

and this will hold if and only if

$$A_{kj} = \frac{<\alpha, \varphi_{kj}>}{<\varphi_{kj}, \varphi_{kj}>} \text{ for} k = 0, 1, \ldots \text{ and } j = 1, 2, \ldots.$$

Condition (4) will hold if and only if

$$\beta = \sum_{k=1}^{\infty}\sum_{j=1}^{\infty}B_{kj}\sqrt{\lambda_{kj}}c\varphi_{kj}$$

and this will hold if and only if

$$B_{kj} = \frac{1}{\sqrt{\lambda_{kj}}c} \frac{< \beta, \varphi_{kj} >}{< \varphi_{kj}, \varphi_{kj} >} \text{ for } k = 1, 2, \ldots \text{ and } j = 1, 2, \ldots.$$

Thus the solution is given by

$$u(x, y, t) = \sum_{k=1}^{\infty} \sum_{j=1}^{\infty} \sin \frac{k\pi x}{L} \sin \frac{j\pi y}{H} \left(A_{kj} \cos \sqrt{\lambda_{kj}} ct + B_{kj} \sin \sqrt{\lambda_{kj}} ct \right)$$

where

$$A_{kj} = \frac{4}{LH} \int_0^L \int_0^H \alpha(x, y) \sin \frac{k\pi x}{L} \sin \frac{j\pi y}{H} dydx \text{ for } k = 1, 2, \ldots \text{and } j = 1, 2, \ldots$$

and

$$B_k = \frac{4}{LH\sqrt{\lambda_{kj}}c} \int_0^L \int_0^H \beta(x, y) \sin \frac{k\pi x}{L} \sin \frac{j\pi y}{H} dydx \text{ for } k = 1, 2, \ldots \text{and } j = 1, 2, \ldots$$

in which

$$\lambda_{kj} = (\frac{k\pi}{L})^2 + (\frac{j\pi}{H})^2.$$