Section 3.3 The Sine Series and The Cosine Series

__Definition__: Suppose that f is integrable on $[0, L]$. The __sine series__ for f is the sequence of functions $\{S_n\}_{n=1}^{\infty}$ given by

$$S_n(x) = \sum_{k=1}^{n} B_k \sin\frac{k\pi x}{L} \quad \text{for \underline{all} } x$$

where

$$B_k = \frac{2}{L}\int_0^L f(x)\sin\frac{k\pi x}{L}\,dx.$$

The __cosine series__ for f is the sequence of functions $\{S_n\}_{n=1}^{\infty}$ given by

$$S_n(x) = A_0 + \sum_{k=1}^{n} A_k \cos\frac{k\pi x}{L} \quad \text{for \underline{all} } x$$

where $A_0 = \frac{1}{L}\int_0^L f(x)\,dx$ and

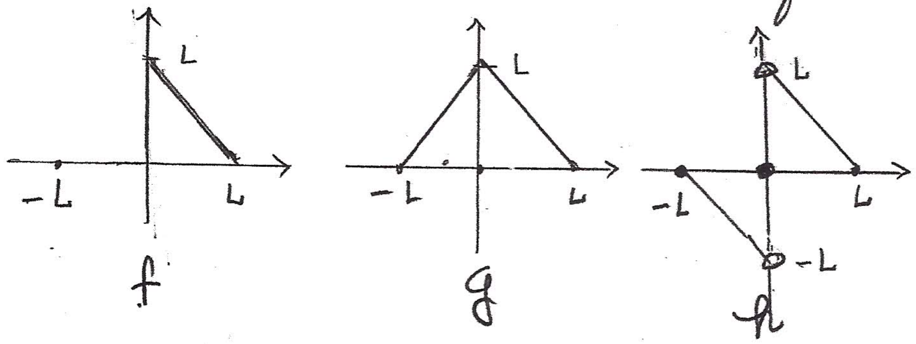
$$A_k = \frac{2}{L}\int_0^L f(x)\cos\frac{k\pi x}{L}\,dx.$$

<u>Definition</u>: Suppose that f is a function defined on $[0,L]$ or on a subset D of $[0,L]$. The <u>even</u> $[-L,L]$ <u>extension</u> of f is the function g given by

$$g(x) = \begin{cases} f(x) & \text{if } 0 \le x \le L \text{ and } x \text{ is in } D \\ f(-x) & \text{if } -L \le x \le 0 \text{ and } -x \text{ is in } D \end{cases}$$

The <u>odd</u> $[-L,L]$ <u>extension</u> of f is the function h given by

$$h(x) = \begin{cases} f(x) & \text{if } 0 < x \le L \text{ and } x \text{ is in } D \\ 0 & \text{if } x = 0 \\ -f(-x) & \text{if } -L \le x < 0 \text{ and } -x \text{ is in } D \end{cases}$$



See also Figures 3.3.1 and 3.3.2 in the text.

Theorem: Suppose that f is an integrable function defined on $[0,L]$. The Sine series for f is the same as the Fourier Series for the odd $[-L,L]$-extension of f and the cosine series for f is the same as the Fourier Series for the even $[-L,L]$ extension of f.

Proof: Let g be the odd $[-L,L]$ extension of f. Since g is odd, the Fourier Series for g is $\{S_n\}$ where

$$S_n(x) = \sum_{k=1}^{n} B_k \sin \frac{k\pi x}{L}$$

$$\text{where } B_k = \frac{1}{L}\int_{-L}^{L} g(x) \sin \frac{k\pi x}{L}\,dx$$

Since g is odd, B_k is also given by

$$B_k = \frac{2}{L}\int_0^L g(x) \sin \frac{k\pi x}{L}\,dx$$

Since $g(x) = f(x)$ for $0 < x \leq L$,

$$B_k = \frac{2}{L}\int_0^L f(x) \sin \frac{k\pi x}{L}\,dx$$

Thus $\{s_n\}$ is also the sine series for f. The proof of the second assertion in the theorem is similar

Example: Let $f(x) = 2 - x$ for $0 \leq x \leq 1$. Sketch the graphs of the limit of the sine series for f and the cosine series for f over the interval $[-5, 5]$

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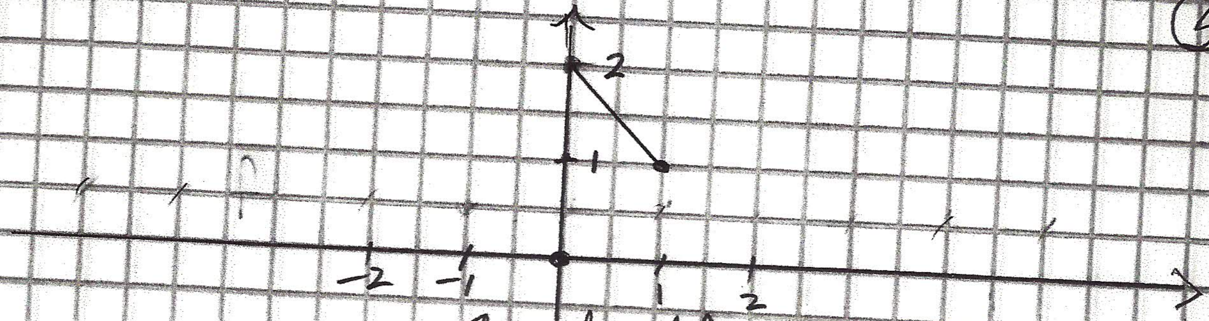
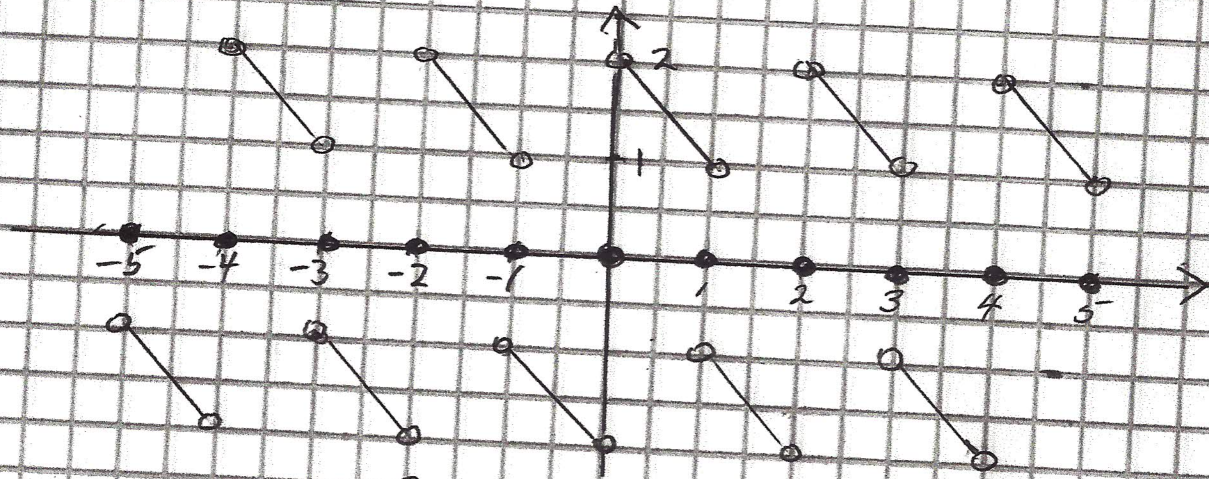
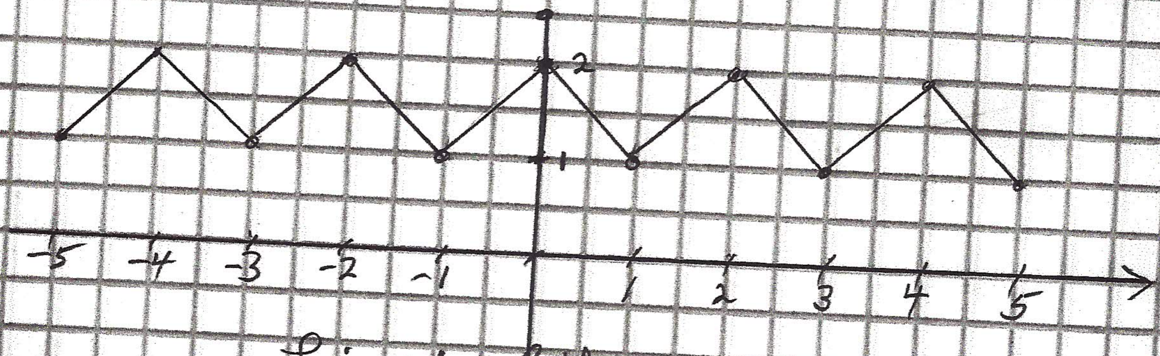


graph of f



Limit of the sine series



Limit of the cosine series