Three by Three Constant Coefficient Homogeneous Systems

Suppose that A is a real 3×3 constant matrix. We consider the differential equation

$$x' = Ax \tag{1}$$

over some interval J. Recall that the characteristic polynomial for A is the function $\mathcal{P}$ given by

$$\mathcal{P}(\lambda) = \det(A - \lambda I)$$

and that the zeros of $\mathcal{P}$ (or roots of the equation $\mathcal{P}(\lambda) = 0$) are the eigenvalues of A. If λ_0 is an eigenvalue then K is a corresponding eigenvector if and only if K is a nonzero three dimensional column vector satisfying

$$(A - \lambda_0 I)K = 0.$$

- **Case I.** If $\mathcal{P}(\lambda) = -(\lambda - \lambda_1)(\lambda - \lambda_2)(\lambda - \lambda_3)$ with $\lambda_i \neq \lambda_j$ when $i \neq j$ so that A has three distinct real eigenvalues λ_1, λ_2, and λ_3, let K_i be an eigenvalue for A corresponding to λ_i for $i = 1, 2, 3$,

  $$x_1(t) = e^{\lambda_1 t} K_1,\ x_2(t) = e^{\lambda_2 t} K_2, \text{ and } x_3(t) = e^{\lambda_3 t} K_3$$

  for all t in J.

- **Case II.** If $\mathcal{P}(\lambda) = -(\lambda - \lambda_1)(\lambda - (\alpha + \beta i))(\lambda - (\alpha - \beta i))$ where each of λ_1, α, and β is a real number with $\beta \neq 0$ so that A has a real eigenvalue λ_1 and complex conjugate eigenvalues $\alpha + \beta i$ and $\alpha - \beta i$, let K_1 be an eigenvector corresponding λ_1 and K_2 be an eigenvector corresponding to $\alpha + \beta i$. Then let

  $$x_1(t) = e^{\lambda_1 t} K_1,\ x_2(t) = \operatorname{Re}\left(e^{(\alpha + \beta i)t} K_2\right) \text{ and } x_3(t) = \operatorname{Im}\left(e^{(\alpha + \beta i)t} K_2\right)$$

  for all t in J.

- **Case III(a).** If $\mathcal{P}(\lambda) = -(\lambda - \lambda_1)(\lambda - \lambda_2)^2$ where $\lambda_1$and λ_2 are distinct real numbers so that A has an eigenvalue λ_1 of algebraic multiplicity 1 and an eigenvalue λ_2 of algebraic multiplicity 2, and

  $$\operatorname{rank}(A - \lambda_2 I) = 1,$$

(This happens if and only if a row-echelon form of $(A - \lambda_2 I)$ has exactly two all zero rows.) let K_1 be an eigenvector corresponding to λ_1 and let K_2 and K_3 be independent eigenvectors corresponding to λ_2. Then let

$$x_1(t) = e^{\lambda_1 t} K_1,\ x_2(t) = e^{\lambda_2 t} K_2, \text{ and } x_3(t) = e^{\lambda_2 t} K_3$$

for all t in J.

- **Case III(b).** If $\mathcal{P}(\lambda) = -(\lambda - \lambda_1)(\lambda - \lambda_2)^2$ where $\lambda_1$and λ_2 are distinct real numbers so that A has an eigenvalue λ_1 of algebraic multiplicity 1 and an eigenvalue λ_2 of algebraic multiplicity 2, and

$$\text{rank}(A - \lambda_2 I) = 2,$$

(This happens if and only if a row-echelon form of $(A - \lambda_2 I)$ has exactly one all zero rows.) let K_1 be an eigenvector corresponding to λ_1, K_2 be an eigenvector corresponding to λ_2 and let W be a three-dimensional column vector satisfying

$$(A - \lambda_2 I)\, W = K_2.$$

Then let

$$x_1(t) = e^{\lambda_1 t} K_1,\ x_2(t) = e^{\lambda_2 t} K_2, \text{ and } x_3(t) = te^{\lambda_2 t} K_2 + e^{\lambda_2 t} W$$

for all t in J.

- **Case IV(a).** If $\mathcal{P}(\lambda) = -(\lambda - \lambda_0)^3$ where λ_0 is a real number so that A has only one eigenvalue λ_0 of algebraic multiplicity 3 and

$$\text{rank}(A - \lambda_0 I) = 0$$

(This happens if and only if $A = \begin{pmatrix} \lambda_0 & 0 & 0 \\ 0 & \lambda_0 & 0 \\ 0 & 0 & \lambda_0 \end{pmatrix}$.), let

$$x_1(t) = e^{\lambda_0 t} \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix},\ x_2(t) = e^{\lambda_0 t} \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \text{ and } x_3(t) = e^{\lambda_0 t} \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

for all t in J.

- **Case IV(b).** If $\mathcal{P}(\lambda) = -(\lambda - \lambda_0)^3$ where λ_0 is a real number so that A has only one eigenvalue λ_0 of algebraic multiplicity 3 and

$$\text{rank}(A - \lambda_0 I) = 1,$$

(This happens if and only if a row-echelon form of $(A - \lambda_2 I)$ has exactly two all zero rows.) let K_1, K_2,and P be such that K_1 and K_2 are linearly independent eigenvectors corresponding to λ_0 and

$$(A - \lambda_0 I)P = K_2.$$

Then let

$$x_1(t) = e^{\lambda_0 t} K_1,\ x_2(t) = e^{\lambda_0 t} K_2, \text{ and } x_3(t) = te^{\lambda_0 t} K_2 + e^{\lambda_0 t} P$$

- **Case IV(c).** If $\mathcal{P}(\lambda) = -(\lambda - \lambda_0)^3$ where λ_0 is a real number so that A has only one eigenvalue λ_0 of algebraic multiplicity 3 and

  $$\text{rank}(A - \lambda_0 I) = 2,$$

  (This happens if and only if a row-echelon form of $(A - \lambda_2 I)$ has exactly two all zero rows.) let K be an eigenvalue corresponding to λ_0, W be a three dimensional constant column vector satisfying

  $$(A - \lambda_0 I)W = K,$$

  and Z be a three dimensional constant column vector satisfying

  $$(A - \lambda_0 I)Z = W.$$

  Then let

  $$x_1(t) = e^{\lambda_0 t}K,\ x_2(t) = te^{\lambda_0 t}K + e^{\lambda_0 t}W, \text{ and } x_3(t) = \frac{t^2}{2}e^{\lambda_0 t}K + te^{\lambda_0 t}W + e^{\lambda_0 t}Z$$

In each case (x_1, x_2, x_3) is a fundamental triple for equation (1) and x is a solution to equation (1) if and only if

$$x(t) = c_1 x_1(t) + c_2 x_2(t) + c_3 x_3(t)$$

for some triple of numbers (c_1, c_2, c_3) and all t in J.