Two by Two Constant Coefficient Homogeneous Systems

Suppose that A is a real 2×2 constant matrix. We consider the differential equation

$$x' = Ax \tag{1}$$

over some interval J. Recall that the characteristic polynomial for A is the function $\mathcal{P}$ given by

$$\mathcal{P}(\lambda) = \det(A - \lambda I)$$

and that the zeros of $\mathcal{P}$ (or roots of the equation $\mathcal{P}(\lambda) = 0$) are the eigenvalues of A. If λ_0 is an eigenvalue then K is a corresponding eigenvector if and only if K is a nonzero two dimensional column vector satisfying

$$(A - \lambda_0 I)K = 0.$$

- **Case I.** If A has two real eigenvalues λ_1 and λ_2 with $\lambda_1 \neq \lambda_2$, let K_1 be an eigenvalue for A corresponding to λ_1, K_2 be an eigenvalue for A corresponding to λ_2,

  $$x_1(t) = e^{\lambda_1 t} K_1 \text{ and } x_2(t) = e^{\lambda_2 t} K_2$$

  for all t in J.

- **Case II.** If A has complex conjugate eigenvalues $\alpha + \beta i$ and $\alpha - \beta i$ where each of α and β is real and $\beta \neq 0$, let K be an eigenvector corresponding to $\alpha + \beta i$,

  $$x_1(t) = \text{Re}\left(e^{(\alpha+\beta i)t} K\right) \text{ and } x_2(t) = \text{Im}\left(e^{(\alpha+\beta i)t} K\right)$$

  for all t in J.

- **Case III(a).** If $\mathcal{P}(\lambda) = (\lambda - \lambda_0)^2$ so that A has only one eigenvalue λ_0 and

  $$\text{rank}(A - \lambda_0 I) = 0$$

  (This happens if and only if $A = \begin{pmatrix} \lambda_0 & 0 \\ 0 & \lambda_0 \end{pmatrix}$.) let

  $$x_1(t) = e^{\lambda_0 t} \begin{pmatrix} 1 \\ 0 \end{pmatrix} \text{ and } x_2(t) = e^{\lambda_0 t} \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

  for all t in J.

- **Case III(b).** If $\mathcal{P}(\lambda) = (\lambda - \lambda_0)^2$ so that A has only one eigenvalue λ_0 and
$$\text{rank}(A - \lambda_0 I) = 1,$$
(This happens if and only if $A \neq \begin{pmatrix} \lambda_0 & 0 \\ 0 & \lambda_0 \end{pmatrix}$.) let K be an eigenvector corresponding to λ_0, and let W be a two-dimensional column vector satisfying
$$(A - \lambda_0 I)\, W = K.$$
Then let
$$x_1(t) = e^{\lambda_0 t} K \text{ and } x_2(t) = t e^{\lambda_0 t} K + e^{\lambda_0 t} W$$
for all t in J.

In each case (x_1, x_2) is a fundamental pair for eqation (1) and x is a solution to equation (1) if and only if

$$x(t) = c_1 x_1(t) + c_2 x_2(t)$$

for some pair of numbers (c_1, c_2) and all t in J.