## Standing Wave Solutions

We have seen that the solution to the wave equation when each end is fixed at zero is given by

$$u(x,t) = \sum_{k=1}^{\infty} u_k(x,t)$$

where

$$u_k(x,t) = (A_k \cos\frac{k\pi ct}{L} + B_k \sin\frac{k\pi ct}{L})(\sin\frac{k\pi x}{L}).$$

These functions $u_k$ are also given by

$$u_k(x,t) = \mathcal{A}_k \sin(\frac{k\pi c}{L}t + \theta_k)\sin\frac{k\pi x}{L}.$$

To get the connection, note that

$$\sin(\frac{k\pi c}{L}t + \theta_k) = \sin\frac{k\pi c}{L}t\cos\theta_k + \cos\frac{k\pi c}{L}t\sin\theta_k$$

so

$$\begin{aligned}\mathcal{A}_k \sin(\frac{k\pi c}{L}t + \theta_k) &= \mathcal{A}_k \sin\theta_k \cos\frac{k\pi c}{L}t + \mathcal{A}_k \cos\theta_k \sin\frac{k\pi c}{L}t \\ &= A_k \cos\frac{k\pi c}{L}t + B_k \sin\frac{k\pi c}{L}t\end{aligned}$$

when

$$A_k = \mathcal{A}_k \sin\theta_k \text{ and } B_k = \mathcal{A}_k \cos\theta_k$$

These last two equations will hold if

$$\mathcal{A}_k = \sqrt{A_k^2 + B_k^2}$$

and

$$\tan\theta_k = \frac{A_k}{B_k}.$$

The solutions $u_k$ are called standing wave or nodal solutions. See the diagrams on the left side of page 141 of the text. Since

$$-1 \le \sin(\frac{k\pi c}{L}t + \theta_k) \le 1$$

when the solution is given by a single $u_k$, the string moves between the graph where $y = -\mathcal{A}_k \sin\frac{k\pi x}{L}$ and the graph where $y = \mathcal{A}_k \sin\frac{k\pi x}{L}$.

The period associated with $u_k$ is

$$2\pi/(\frac{k\pi c}{L}) = \frac{2L}{kc}$$

and the frequency is

$$\frac{kc}{2L}.$$

The fundamental frequency is $\frac{c}{2L}$ and the harmonic frequencies are the positive integral multiples of the fundamental frequency.