An Equilibrium Temperature Distrubution Problem

PROBLEM: Find the value of β for which the following problem has an equilibrium temperature distribution.

$$\frac{\partial w}{\partial t}(x,t) = \frac{\partial^2 w}{\partial x^2}(x,t) + x \text{ for } t \geq 0 \text{ and } 0 \leq x \leq L,$$

$$w(x,0) = f(x) \text{ for } 0 \leq x \leq L,$$

$$\frac{\partial w}{\partial x}(0,t) = 1, \text{ and } \frac{\partial w}{\partial x}(L,t) = \beta \text{ for } t \geq 0.$$

Let u be the equilibrium solution so that

$$u(x) = \lim_{t\to\infty} w(x,t) \text{ for } 0 \leq x \leq L.$$

Find a formula for $u(x)$ that does not contain any undetermined constants.

SOLUTION: The problem for the equilibrium distribution u is obtained by setting the time derivative equal to zero in the PDE for w and replacing $w(x,t)$ with $u(x)$. Thus

$$0 = u''(x) + x \text{ or } u''(x) = -x \text{ for } 0 \leq x \leq L,$$

$$u'(0) = 1, \text{ and } u'(L) = \beta.$$

Integrating once, we have from the ODE for u that

$$u'(x) = -\frac{1}{2}x^2 + c_1.$$

Applying $u'(0) = 1$ we have

$$1 = -\frac{1}{2}0^2 + c_1.$$

So $c_1 = 1$ and

$$u'(x) = -\frac{1}{2}x^2 + 1.$$

Applying $u'(L) = \beta$ we have

$$\beta = -\frac{1}{2}L^2 + 1.$$

Integrating each side of the last DE for $u'(x)$ we have

$$u(x) = -\frac{1}{6}x^3 + x + c_2.$$

To complete the solution, we need to find c_2.

Returning to the PDE for w we have

$$\frac{d}{dt}\int_0^L w(x,t)dx = \int_0^L \frac{\partial w(x,t)}{\partial t}dx = \int_0^L (\frac{\partial^2 w}{\partial x^2}(x,t) + x)dx.$$

Using the Fundamental Theorem fo Calculus, we get

$$\frac{d}{dt}\int_0^L w(x,t)dx = \frac{\partial w(L,t)}{\partial x} - \frac{\partial w(0,t)}{\partial x} + \frac{1}{2}L^2.$$

So

$$\frac{d}{dt}\int_0^L w(x,t)dx = \beta - 1 + \frac{1}{2}L^2.$$

Since $\beta = -\frac{1}{2}L^2 + 1$,

$$\frac{d}{dt}\int_0^L w(x,t)dx = 0.$$

Thus

$$\int_0^L w(x,t)dx$$

is constant in time, and we get

$$\int_0^L w(x,0)dx = \lim_{t\to\infty}\int_0^L w(x,t)dx = \int_0^L \lim_{t\to\infty} w(x,t)dx.$$

From this we may conclude that

$$\int_0^L f(x)dx = \int_0^L u(x)dx,$$

so

$$\int_0^L f(x)dx = \int_0^L (-\frac{1}{6}x^3 + x + c_2)dx.$$

Evaluating the integral on the right we have

$$\int_0^L f(x)dx = -\frac{1}{24}L^4 + \frac{1}{2}L^2 + c_2 L.$$

Thus

$$c_2 = \frac{1}{24}L^3 - \frac{1}{2}L + \frac{1}{L}\int_0^L f(x)dx.$$

Finally

$$u\left(x\right) = -\frac{1}{6}x^3 + x + \frac{1}{24}L^3 - \frac{1}{2}L + \frac{1}{L}\int_0^L f(x)dx.$$