Laplace Equation Problem VI

PROBLEM: Suppose that $0 < r_1 < r_2$ and each of f_1 and f_2 is a function defined on $[-\pi,\pi]$. Find the solution u to Laplace's equation in polar coordinates

$$\frac{\partial^2 u}{\partial r^2}(r,\theta) + \frac{1}{r}\frac{\partial u}{\partial r}(r,\theta) + \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2}(r,\theta) = 0 \tag{1}$$

in the annulus consisting of all (r,θ) where

$$r_1 \le r \le r_2 \text{ and } -\pi \le \theta \le \pi$$

subject to

$$\begin{aligned} u(r,-\pi) &= u(r,\pi) \text{ for } r_1 \le r \le r_2, & (2)\\ \frac{\partial u}{\partial \theta}(r,-\pi) &= \frac{\partial u}{\partial \theta}(r,\pi) \text{ for } r_1 \le r \le r_2, & (3)\\ u(r_1,\theta) &= f_1(\theta) \text{ for } -\pi \le \theta \le \pi, \text{ and} & (4)\\ u(r_2,\theta) &= f_2(\theta) \text{ for } -\pi \le \theta \le \pi. & (5) \end{aligned}$$

Then find the solution in case $r_1 = 3$, $r_2 = 6$, $f_1(\theta) = 5\cos\theta$, and $f_2(\theta) = 10\sin\theta$.

SOLUTION: Suppose that u is an elementary separated solution to (1). This means

$$u(r,\theta) = \varphi(\theta)G(r)$$

for some pair of one-place functions φ and G. Inserting this into (1), we have

$$\varphi(\theta)G''(r) + \frac{1}{r}\varphi(\theta)G'(r) + \frac{1}{r^2}\varphi''(\theta)G(r) = 0 \tag{6}$$

Assuming for now that

$$u(r,\theta) \ne 0,$$

and dividing each side of (6) by

$$\frac{\varphi(\theta)G(r)}{r^2}$$

we have

$$r^2\frac{G''(r)}{G(r)} + r\frac{G'(r)}{G(r)} = -\frac{\varphi''(\theta)}{\varphi(\theta)}$$

This holds for all r with $r_1 \le r \le r_2$ and θ with $-\pi \le \theta \le \pi$; so there is a constant λ such that

$$r^2\frac{G''(r)}{G(r)} + r\frac{G'(r)}{G(r)} = -\frac{\varphi''(\theta)}{\varphi(\theta)} = \lambda \tag{7}$$

for all r with $r_1 \le r \le r_2$ and θ with $-\pi \le \theta \le \pi$. From (7) we then have

$$-\varphi''(\theta) = \lambda\varphi(\theta) \text{ for all } \theta \text{ in } [-\pi,\pi] \tag{8}$$

and

$$r^2G''(r) + rG'(r) - \lambda G(r) = 0 \text{ for all } r \text{ in } [r_1,r_2]. \tag{9}$$

It is worth noting that if

$$u(r,\theta) = \varphi(\theta)G(r)$$

and (8) and (9) hold, then

$$\begin{aligned}
\frac{\partial^2 u}{\partial r^2}(r,\theta) + \frac{1}{r}\frac{\partial u}{\partial r}(r,\theta) + \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2}(r,\theta) &= \varphi(\theta)G''(r) + \frac{1}{r}\varphi(\theta)G'(r) + \frac{1}{r^2}\varphi''(\theta)G(r) \\
&= \varphi(\theta)\left(G''(r) + \frac{1}{r}G'(r)\right) + \left(\frac{1}{r^2}\varphi''(\theta)G(r)\right) \\
&= \varphi(\theta)\lambda\frac{1}{r^2}G(r) - \frac{1}{r^2}\lambda\varphi(\theta)G(r) \\
&= 0
\end{aligned}$$

so the PDE (1) will be satisfied, and we no longer need to assume that $u(r,\theta) \neq 0$.

Continuing with our assumption that

$$u(r,\theta) = \varphi(\theta)G(r)$$

we have from conditions (2) and (3) which stated that

$$u(r,-\pi) = u(r,\pi) \text{ for } r_1 \leq r \leq r_2$$

and

$$\frac{\partial u}{\partial \theta}(r,-\pi) = \frac{\partial u}{\partial \theta}(r,\pi) \text{ for } r_1 \leq r \leq r_2,$$

that either $G(r) = 0$ for all r in $[r_1, r_2]$ which we reject because of (3) and (4) (which stated that $u(r_1,\theta) = f_1(\theta)$ and $u(r_2,\theta) = f_2(\theta)$ for $-\pi \leq \theta \leq \pi$) or

$$\varphi(-\pi) = \varphi(\pi) \tag{10}$$

and

$$\varphi'(-\pi) = \varphi'(\pi) \tag{11}$$

which we then must accept.

The two-point boundary value problem consisting of (8), (10), and (11) (which we repeat here)

$$\begin{aligned}
-\varphi'' &= \lambda\varphi \text{ on } [-\pi,\pi] \\
\varphi(-\pi) &= \varphi(\pi), \text{ and} \\
\varphi'(-\pi) &= \varphi'(\pi)
\end{aligned}$$

is one which we have studied. A proper listing of eigenvalues and eigenfunctions for this problem is $\{\lambda_k\}_{k=0}^{\infty}$ and $\{\varphi_k\}_{k=0}^{\infty}$ where

$$\lambda_0 = 0, \ \varphi_0(\theta) = 1 \text{ for } -\pi \leq \theta \leq \pi,$$

$$\begin{aligned}
\lambda_{2k-1} &= \lambda_{2k} = k^2 \text{ for } k = 1,2,3,\ldots, \\
\varphi_{2k-1}(\theta) &= \cos k\theta, \text{ and } \varphi_{2k}(\theta) = \sin k\theta \text{ for } k = 1,2,3,\ldots \text{ and } -\pi \leq \theta \leq \pi.
\end{aligned}$$

Equation (9) is a Cauchy-Euler equation. When $\lambda = 0$ then

$$G(r) = c_1 + c_2 \ln r,$$

and when k is a positive integer and $\lambda = k^2$ then

$$G(r) = c_1 r^k + c_2 r^{-k}.$$

Considering the possible combinations, we expect the solution to (1)-(5) to be of the form

$$u(r,\theta) = A_0 + B_0 \ln r + \sum_{k=1}^{\infty} \left[\left(A_k r^k + B_k r^{-k} \right) \cos k\theta + \left(C_k r^k + D_k r^{-k} \right) \sin k\theta \right]. \tag{12}$$

for $r_1 \le r \le r_2$ and $-\pi \le \theta \le \pi$.

In order that (4) and (5) hold it is necessary and sufficient that

$$f_1(\theta) = A_0 + B_0 \ln r_1 + \sum_{k=1}^{\infty} \left[\left(A_k r_1^k + B_k r_1^{-k} \right) \cos k\theta + \left(C_k r_1^k + D_k r_1^{-k} \right) \sin k\theta \right] \tag{13}$$

and

$$f_2(\theta) = A_0 + B_0 \ln r_2 + \sum_{k=1}^{\infty} \left[\left(A_k r_2^k + B_k r_2^{-k} \right) \cos k\theta + \left(C_k r_2^k + D_k r_2^{-k} \right) \sin k\theta \right] \tag{14}$$

for $-\pi \le \theta \le \pi$. Equations (13) and (14) express $f_1$and f_2 as limits of Fourier series. Thus

$$A_0 + B_0 \ln r_j = \frac{1}{2\pi} \int_{-\pi}^{\pi} f_j(\theta) d\theta \text{ for } j = 1 \text{ and } j = 2, \tag{15}$$

$$A_k r_j^k + B_k r_j^{-k} = \frac{1}{\pi} \int_{-\pi}^{\pi} f_j(\theta) \cos k\theta d\theta \text{ for } j = 1 \text{ and } j = 2, \tag{16}$$

and

$$C_k r_j^k + D_k r_j^{-k} = \frac{1}{\pi} \int_{-\pi}^{\pi} f_j(\theta) \sin k\theta d\theta \text{ for } j = 1 \text{ and } j = 2. \tag{17}$$

Equations (15) give two linear equations in the unknowns A_0 and B_0; equations (16) give two linear equations in the unknowns A_k and B_k for $k = 1, 2, \ldots$; and equations (17) give two linear equations in the unknowns C_k and D_k for $k = 1, 2, \ldots$. We have $r_1 \ne r_2$; so in each case, the coefficient matrix is nonsingular. If the functions f_1 and f_2 are reasonable, the solution to (1)-(5) is given by (12) where the coefficients are determined by (15)-(17).

Example 1 *Find the solution in case* $r_1 = 3$*,* $r_2 = 6$*,* $f_1(\theta) = 5\cos\theta$*, and* $f_2(\theta) = 10\sin\theta$*.*

Solution: The functions $\{\varphi_k\}$ are orthogonal on $[-\pi, \pi]$, $< \varphi_0, \varphi_0 >= 2\pi$, $< \varphi_k, \varphi_k >= \pi$ for $k = 1, 2, \ldots$, $f_1 = 5\varphi_1$, and $f_2 = 10\varphi_2$. Thus

$$A_0 + B_0 \ln r_j = 0 \text{ for } j = 1 \text{ and } j = 2,$$

$$A_1 r_1^1 + B_1 r_1^{-1} = 5,$$

$$A_1 r_2^1 + B_1 r_2^{-1} = 0,$$
$$A_k r_j^k + B_k r_j^{-k} = 0 \text{ for } j = 1 \text{ and } j = 2 \text{ and } k = 2, 3, \ldots,$$
$$C_1 r_1^1 + D_1 r_1^{-1} = 0,$$
$$C_1 r_2^1 + D_1 r_2^{-1} = 10,$$

and

$$C_k r_j^k + D_k r_j^{-k} = 0 \text{ for } j = 1 \text{ and } j = 2 \text{ and } k = 2, 3, \ldots.$$

From these it follows that

$$\begin{aligned} A_1 &= -\frac{5}{9}, \\ B_1 &= 20, \\ C_1 &= \frac{20}{9}, \\ D_1 &= -20, \end{aligned}$$

and all other coefficients are zero. Thus

$$u(r,\theta) = \left(-\frac{5r}{9} + \frac{20}{r}\right)\cos\theta + \left(\frac{20r}{9} - \frac{20}{r}\right)\sin\theta.$$