Laplace's Equation in a Disk

Note. The solution to Laplace's euqation in polar coordiates in an anulus,

$$u(r,\theta) = A_0 + B_0 \ln r + \sum_{k=1}^{\infty} \left[\left(A_k r^k + B_k r^{-k} \right) \cos k\theta + \left(C_k r^k + D_k r^{-k} \right) \sin k\theta \right],$$

can be split into two pieces,

$$A_0 + \sum_{k=1}^{\infty} \left[A_k r^k \cos k\theta + C_k r^k \sin k\theta \right]$$

and

$$B_0 \ln r + \sum_{k=1}^{\infty} \left[B_k r^{-k} \cos k\theta + D_k r^{-k} \sin k\theta \right].$$

the first sum exists for $r < r_2$ and the second sum exists for $r > r_1$. Thus the solution to Laplace's equation in polar coordinates in a circular disk when the solution is specified on the boundary,

$$\frac{\partial^2 u}{\partial r^2}(r,\theta) + \frac{1}{r}\frac{\partial u}{\partial r}(r,\theta) + \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2}(r,\theta) = 0$$

and

$$u(a,\theta) = f(\theta)$$

for $0 \le r \le a$ and $-\pi \le \theta \le \pi$ is given by

$$u(r,\theta) = A_0 + \sum_{k=1}^{\infty} \left[A_k r^k \cos k\theta + B_k r^k \sin k\theta \right]$$

where

$$A_0 = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(\theta) d\theta,$$

$$A_k = \frac{1}{a^k \pi} \int_{-\pi}^{\pi} f(\theta) \cos k\theta d\theta,$$

and

$$B_k = \frac{1}{a^k \pi} \int_{-\pi}^{\pi} f(\theta) \sin k\theta d\theta$$

for $k = 1, 2, \ldots$.