A METHOD FOR SOLVING NONHOMOGENEOUS DIFFERENTIAL EQUATIONS

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We give a formula that can be used to solve any regular nth order nonhomogenous linear differential equation with a continuous right side provided that a fundamental sequence (i.e. a linearly independent n-tuple of solutions to the corresponding homogeneous equation) and certain antiderivatives can be found. We begin with the second order case.

Theorem 1. *Suppose that L is a second order regular linear differential operator over the interval J with*

$$Ly(x) = a(x)y''(x) + b(x)y'(x) + c(x)y(x).$$

Suppose that (u_1, u_2) is a fundamental pair for L and that f is a continuous function defined on J. Let w be the Wronskian of (u_1, u_2), let I_1 be an antiderivative of $\frac{u_1 f}{aw}$,

$$I_1(x) = \int \frac{u_1(x)f(x)}{a(x)w(x)} dx,$$

and let I_2 be an antiderivative of $\frac{u_2 f}{aw}$,

$$I_2(x) = \int \frac{u_2(x)f(x)}{a(x)w(x)} dx.$$

Finally, let v be given by

$$v = u_2 I_1 - u_1 I_2.$$

It follows that

$$Lv = f.$$

It is worth noting that v can be given on one line by the following formula.

$$v(x) = u_2(x) \int \frac{u_1(x)f(x)}{a(x)w(x)} dx - u_1(x) \int \frac{u_2(x)f(x)}{a(x)w(x)} dx$$

Only one antiderivative is needed in each case, so leave off the "$+C$" when finding the integrals.

Proof. Let v be as indicated. Then

$$\begin{aligned} v' &= u_2' I_1 + u_2 I_1' - u_1' I_2 - u_1 I_2' \\ &= u_2' I_1 + u_2 \frac{u_1 f}{aw} - u_1' I_2 - u_1 \frac{u_2 f}{aw} \\ &= u_2' I_1 - u_1' I_2 \end{aligned}$$

and since $w = u_1 u_2' - u_1' u_2$,

$$\begin{aligned} v'' &= u_2'' I_1 + u_2' I_1' - u_1'' I_2 - u_1' I_2' \\ &= u_2'' I_1 + u_2' \frac{u_1 f}{aw} - u_1'' I_2 - u_1' \frac{u_2 f}{aw} \\ &= u_2'' I_1 - u_1'' I_2 + \frac{(u_1 u_2' - u_1' u_2) f}{aw} \\ &= u_2'' I_1 - u_1'' I_2 + \frac{f}{a}. \end{aligned}$$

Thus

$$\begin{aligned} Lv &= av'' + bv' + cv \\ &= a\left(u_2'' I_1 - u_1'' I_2 + \frac{f}{a}\right) \\ &\quad + b\left(u_2' I_1 - u_1' I_2\right) \\ &\quad + c\left(u_2 I_1 - u_1 I_2\right). \end{aligned}$$

So

$$\begin{aligned} Lv &= (au_2'' + bu_2' + cu_2) I_1 - (au_1'' + bu_1' + cu_1) I_2 + f \\ &= (Lu_2) I_1 - (Lu_1) I_2 + f \\ &= 0 \cdot I_1 - 0 \cdot I_2 + f \\ &= f. \end{aligned}$$

□

Example 1. Find all functions y such that

$$y''(x) + y(x) = \tan x \text{ for } 0 < x < \frac{\pi}{2}.$$

Solution. We are looking for all y such that

$$Ly = f \text{ on } J$$

where $Ly = y'' + y$, $f(x) = \tan x$, and $J = (0, \frac{\pi}{2})$. The leading coefficient function is the constant function with value 1, and a fundamental sequence for L is (u_1, u_2) where $u_1(x) = \cos x$ and $u_2(x) = \sin x$. The Wronskian w is given by

$$w(x) = \det \begin{bmatrix} \cos x & \sin x \\ -\sin x & \cos x \end{bmatrix} = 1.$$

$$I_1(x) = \int \frac{\cos x \tan x}{(1)(1)} dx = -\cos x \text{ and}$$

$$\begin{aligned} I_2(x) &= \int \frac{\sin x \tan x}{(1)(1)} dx = \int \frac{\sin^2 x}{\cos x} dx = \int \frac{1 - \cos^2 x}{\cos x} dx \\ &= \int (\sec x - \cos x) dx = \ln(\sec x + \tan x) - \sin x. \end{aligned}$$

So a function v such that $Lv = f$ is given by

$$\begin{aligned} v(x) &= (\sin x)(-\cos x) - (\cos x)(\ln(\sec x + \tan x) - \sin x) \\ &= -\cos x \ln(\sec x + \tan x). \end{aligned}$$

Since $Ly = f$ if and only if $y = u + v$ for some u such that $Lu = 0$, we find that y is a solution to the problem if and only if

$$y(x) = c_1 \cos x + c_2 \sin x - \cos x \ln(\sec x + \tan x)$$

for some pair of numbers (c_1, c_2) and all x with $0 < x < \dfrac{\pi}{2}$.

Sometimes when the method presented here is used, the function v can be seen to be of the form

$$v = v_1 + v_2$$

where v_2 is a solution to the corresponding homogeneous equation.. In this case v_1 provides a simpler solution than v to the nonhomogeneous equation. This is because

$$Lv_1 = L(v - v_2) = Lv - Lv_2 = f - 0 = f.$$

Example 2. Find all functions y such that

$$y''(x) - 3y'(x) + 2y(x) = \frac{1}{1 + e^{-x}} \text{ for all } x \text{ in } \mathbb{R}.$$

Solution. We are looking for all y such that

$$Ly = f \text{ on } J$$

where $Ly = y'' - 3y' + 2y,\ f(x) = \dfrac{1}{1 + e^{-x}}$, and $J = \mathbb{R}$. The leading coefficient function is the constant function with value 1, and a fundamental sequence for L is (u_1, u_2) where $u_1(x) = e^x$ and $u_2(x) = e^{2x}$. The Wronskian w is given by

$$w(x) = \det \begin{bmatrix} e^x & e^{2x} \\ e^x & 2e^{2x} \end{bmatrix} = e^{3x}.$$

$$I_1(x) = \int \frac{e^x}{(1)(e^{3x})} \cdot \frac{1}{1 + e^{-x}} dx = \ln(1 + e^{-x}) - e^{-x} \text{ and}$$

$$I_2(x) = \int \frac{e^{2x}}{(1)(e^{3x})} \cdot \frac{1}{1 + e^{-x}} dx = -\ln(1 + e^{-x}).$$

So a function v such that $Lv = f$ is given by

$$\begin{aligned} v(x) &= e^{2x}(\ln(1 + e^{-x}) - e^{-x}) + e^x \ln\left(1 + e^{-x}\right) \\ &= e^{2x} \ln(1 + e^{-x}) + e^x \ln(1 + e^{-x}) - e^x \\ &= v_1(x) + v_2(x) \end{aligned}$$

where

$$\begin{aligned} v_1(x) &= e^{2x} \ln(1 + e^{-x}) + e^x \ln(1 + e^{-x}) \text{ and} \\ v_2(x) &= -e^x. \end{aligned}$$

Since $Lv_2 = -Lu_1 = 0$, a simpler solution to the nonhomogeneous equation is v_1. Since $Ly = f$ if and only if $y = u + v_1$ for some u such that $Lu = 0$, we find that y is a solution to the problem if and only if

$$y(x) = c_1 e^x + c_2 e^{2x} + e^{2x} \ln(1 + e^{-x}) + e^x \ln(1 + e^{-x})$$

for some pair of numbers (c_1, c_2) and all x.

The formula that we have given for the second order is a special case of the one presented in the following theorem.

Theorem 2. *Suppose that L is an nth order regular linear differential operator over the interval J with*

$$Ly(x) = a_0(x)y^{(n)}(x) + a_1(x)y^{(n-1)}(x) + \cdots + a_n(x)y(x).$$

Suppose that $(u_1, \ldots, u_n)$ is a fundamental sequence for L and that f is a continuous function defined on J. Let w be the Wronskian of $(u_1, \ldots, u_n)$, and for $k = 1, \ldots, n$, let $w_k$be the determinant of the matrix obtained by replacing the kth column of the Wronski matrix (the matrix used to find the Wronskian) of $(u_1, \ldots, u_n)$ with the column

$$\begin{pmatrix} 0 \\ \vdots \\ 0 \\ 1 \end{pmatrix}.$$

For $k = 1, \ldots, n$, let I_k be an antiderivative of $\dfrac{w_k f}{a_0 w}$,

$$I_k(x) = \int \frac{w_k(x)f(x)}{a_0(x)w(x)}dx.$$

Finally, let v be given by

$$v(x) = \sum_{k=1}^{n} u_k(x)I_k(x).$$

It follows that

$$Lv = f.$$

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