An Equilibrium Temperature Distrubution Problem

PROBLEM: Determine the equilibrium temperature distribution for a one-dimensional rod of length L when the left end is held at temperature T, the right end is insulated , and the source term Q is given by

$$Q(x) = K_0 x^2.$$

Do this by solving

$$K_0 u''(x) + Q(x) = 0 \text{ for } 0 \leq x \leq L$$

subject to

$$u(0) = T \text{ and } u'(L) = 0.$$

SOLUTION: The DE to solve is

$$K_0 u''(x) + K_0 x^2 = 0$$

or

$$u''(x) = -x^2.$$

Integrating once we have

$$u'(x) = -\frac{1}{3}x^3 + c_{1..}$$

The right end point condition tell us that

$$0 = -\frac{1}{3}L^3 + c_1;$$

so

$$c_1 = \frac{1}{3}L^3.$$

Thus

$$u'(x) = -\frac{1}{3}x^3 + \frac{1}{3}L^3.$$

Integrating again we have

$$u(x) = -\frac{1}{12}x^4 + \frac{1}{3}L^3 x + c_2,$$

and using the left end point condition, $u(0) = T$ we have

$$T = -\frac{1}{12}0^4 + \frac{1}{3}L^3 0 + c_2;$$

so $c_2 = T$ and we have

$$u(x) = -\frac{1}{12}x^4 + \frac{1}{3}L^3 x + T.$$