Laplace Equation Problem I

PROBLEM: Derive the solution to

$$\begin{aligned}
\frac{\partial^2 u}{\partial x^2}(x,y) + \frac{\partial^2 u}{\partial y^2}(x.y) &= 0 \text{ for } 0 \le x \le L \text{ and } 0 \le y \le H, &(1)\\
u(0,y) &= 0 \text{ for } 0 \le y \le H, &(2)\\
u(L,y) &= 0 \text{ for } 0 \le y \le H, &(3)\\
u(x,H) &= 0 \text{ for } 0 \le x \le L, \text{ and} &(4)\\
u(x,0) &= f(x) \text{ for } 0 \le x \le L. &(5)
\end{aligned}$$

Where each of H and L is a positive number. Then find the solution when

$$f(x) = x(L-x) \text{ for } 0 \le x \le L. \tag{6}$$

SOLUTION: Suppose that u is an elementary separated solution to (1). This means

$$u(x,y) = \varphi(x)h(y)$$

for some pair of one-place functions φ and h. Inserting this into (1), we have

$$\varphi''(x)h(y) + \varphi(x)h''(y) = 0. \tag{7}$$

Assuming for now that

$$u(x.y) \ne 0,$$

and dividing each side of (7) by $\varphi(x)h(y)$, we have

$$\frac{\varphi''(x)h(y)}{\varphi(x)h(y)} + \frac{\varphi(x)h''(y)}{\varphi(x)h(y)} = 0,$$

so

$$\frac{h''(y)}{h(y)} = -\frac{\varphi''(x)}{\varphi(x)}.$$

This holds for all y with $0 \le y \le H$ and x with $0 \le x \le L$, so there is a constant λ such that

$$\frac{h''(y)}{h(y)} = \lambda = -\frac{\varphi''(x)}{\varphi(x)} \tag{8}$$

for all y with $0 \le y \le H$ and x with $0 \le x \le L$. From (8) we then have

$$-\varphi''(x) = \lambda\varphi(x) \text{ for all } x \text{ in } [0,L] \tag{9}$$

and

$$h''(y) = \lambda h(y) \text{ for all } y \text{ in } [0,H]. \tag{10}$$

It is worth noting that if

$$u(x,y) = \varphi(x)h(y)$$

and (9) and (10) hold, then

$$\begin{aligned}
\frac{\partial^2 u}{\partial x^2}(x,y) &= \varphi''(x)h(y) = -\lambda\varphi(x)h(y)\\
&= -\varphi(x)h''(y) = -\frac{\partial^2 u}{\partial y^2}(x,y)
\end{aligned}$$

so the PDE (1)

$$\frac{\partial^2 u}{\partial x^2}(x,y) + \frac{\partial^2 u}{\partial y^2}(x.y) = 0$$

will be satisfied, and we no longer need to assume that $u(x,y) \neq 0$. Continuing with our assumption that

$$u(x,y) = \varphi(x)h(y)$$

we have from conditions (2) and (3) (which stated that $u(0,y) = 0 = u(L,y)$) that either $h(y) = 0$ for all y in $[0,H]$ which we reject because of (5) (which stated that $u(x,0) = f(x)$) or

$$\varphi(0) = 0 \tag{11}$$

and

$$\varphi(L) = 0 \tag{12}$$

which we must accept. In a similar way we have from (4) (which stated that $u(x,H) = 0$) that

$$h(H) = 0 \tag{13}$$

The Sturm-Louville problem consisting of (9), (11), and (12) (which we repeat here)

$$\begin{aligned} -\varphi'' &= \lambda\varphi \text{ on } [0,L] \\ \varphi(0) &= 0, \text{ and} \\ \varphi(L) &= 0 \end{aligned}$$

is one which we have studied. A proper listing of eigenvalues and eigenfunctions for this problem is

$$\{\lambda_k\}_{k=1}^{\infty} \text{ and } \{\varphi_k\}_{k=1}^{\infty}$$

where

$$\lambda_k = (\frac{k\pi}{L})^2 \text{ for } k = 1, 2, \ldots$$

and

$$\varphi_k(x) = \sin\frac{k\pi}{L}x \text{ for all } x \text{ in } [0,L] \text{ and } k = 1, 2, \ldots.$$

The equation (10)

$$h''(y) = \lambda h(y)$$

is equivalent to

$$h''(y) - \lambda h(y) = 0. \tag{14}$$

When $\lambda > 0$ as it must be because all eigenvalues for the problem (9), (11), and (12) are positive, a familiar linearly independent pair of solutions to (14) is the pair whose values at y are

$$e^{\sqrt{\lambda}y} \text{ and } e^{-\sqrt{\lambda}y}.$$

Another linearly independent pair of solutions is the pair whose values at y are

$$\cosh\sqrt{\lambda}y \text{ and } \sinh\sqrt{\lambda}y.$$

A third linearly independent pair of solutions is the pair whose values at y are

$$\sinh\sqrt{\lambda}y \text{ and } \sinh\sqrt{\lambda}(H-y).$$

This can be verified by direct substitution into the differential equation (14) and use of the Wronskian. We could use any of these three pairs, but the last one makes our solution process easier. Since h is a solution to (14), we have

$$h(y) = c_1\sinh\sqrt{\lambda}y + c_2\sinh\sqrt{\lambda}(H-y).$$

We have from (13) that $h(H)=0$, so

$$c_1\sinh\lambda H + c_2\sinh\sqrt{\lambda}(H-H) = 0,$$

Using the fact that $\sinh 0 = 0$ and $\sinh z \neq 0$ when $z \neq 0$, we have that $c_1 = 0$ and see that when $\lambda = \lambda_k$ then the solutions to (13) and (14) are constant multiples of h_k where

$$h_k(y) = \sinh\sqrt{\lambda_k}(H-y).$$

Let us recall the original problem.

$$\begin{aligned}
\frac{\partial^2 u}{\partial x^2}(x,y) + \frac{\partial^2 u}{\partial y^2}(x.y) &= 0 \text{ for } 0 \le x \le L \text{ and } 0 \le y \le H, && (1)\\
u(0,y) &= 0 \text{ for } 0 \le y \le H, && (2)\\
u(L,y) &= 0 \text{ for } 0 \le y \le H, && (3)\\
u(x,H) &= 0 \text{ for } 0 \le x \le L, \text{ and} && (4)\\
u(x,0) &= f(x) \text{ for } 0 \le x \le L. && (5)
\end{aligned}$$

The problem consisting of (1), (2), (3), and (4) is linear and homogeneous, so if $\{E_k\}_{k=1}^n$is a finite sequence of numbers and

$$u(x,y) = \sum_{k=1}^{n} E_k\varphi_k(x)h_k(y),$$

then u will be a solution to (1), (2), (3), and (4). Thus we hope that the solution to the problem consisting of (1) through (5) will be of the form

$$u(x,y) = \sum_{k=1}^{\infty} E_k\varphi_k(x)h_k(y)$$

for some perhaps infinite sequence of constants $\{E_k\}_{k=1}^{\infty}$.

Condition (5)

$$u(x,0) = f(x) \text{ for } x \text{ in } [0,L],$$

implies

$$f = \sum_{k=1}^{\infty} E_k\varphi_k h_k(0) = \sum_{k=1}^{\infty}(E_k\sinh\sqrt{\lambda_k}H)\varphi_k.$$

Since $\{\varphi_k\}_{k=1}^{\infty}$ is an orthogonal sequence of non zero function this implies

$$(E_k \sinh \sqrt{\lambda_k} H) = \frac{< f, \varphi_k >}{< \varphi_k, \varphi_k >}$$

so

$$E_k = \frac{< f, \varphi_k >}{\sinh \sqrt{\lambda_k} H < \varphi_k, \varphi_k >}$$

for $k = 1, 2, \ldots$where the inner product is defined by

$$< \alpha, \beta >= \int_0^L \alpha(x)\beta(x)dx.$$

For this sequence $\{\varphi_k\}$,

$$< \varphi_k, \varphi_k >= \int_0^L (\sin \frac{k\pi x}{L})^2 dx = \frac{L}{2} \text{ for } k = 1, 2, \ldots.$$

In summary, the solution to the original problem (1) through (5) is u where

$$u(x, y) = \sum_{k=1}^{\infty} E_k \sin \frac{k\pi x}{L} \sinh \frac{k\pi}{L}(H - y)$$

in which

$$E_k = \frac{2}{L \sinh \frac{k\pi H}{L}} \int_0^L f(x) \sin \frac{k\pi x}{L} dx \text{ for } k = 1, 2, \ldots$$

If f is given by

$$f(x) = x(L - x) \text{ for } 0 \leq x \leq L$$

then

$$E_k = \frac{2}{L \sinh \frac{k\pi H}{L}} \int_0^L x(L - x) \sin \frac{k\pi x}{L} dx$$

Remembering that

$$\sin k\pi = 0 \text{ and } \cos k\pi = (-1)^k$$

and integrating by parts twice, we find that

$$E_k = \frac{4L^2}{\pi^3 k^3 \sinh \frac{k\pi H}{L}}(1 - (-1)^k)$$

so

$$u(x, y) = \frac{4L^2}{\pi^3} \sum_{k=1}^{\infty} \frac{1}{k^3 \sinh \frac{k\pi H}{L}}(1 - (-1)^k) \sin \frac{k\pi x}{L} \sinh \frac{k\pi}{L}(H - y).$$