Laplace Equation Problem V

PROBLEM: Suppose that each of L and H is a positive number and that F is a continuous function defined on the boundary of the rectangle consisting of all (x, y) where $0 \leq x \leq L$ and $0 \leq y \leq H$. Find the solution to

$$\frac{\partial^2 u}{\partial x^2}(x, y) + \frac{\partial^2 u}{\partial y^2}(x.y) = 0 \tag{1}$$

for $0 \leq x \leq L$ and $0 \leq y \leq H$ such that

$$u(x, y) = F(x, y) \tag{2}$$

for all (x, y) on the boundary of the rectangle.

SOLUTION: In order for (2) to hold at the corners of the rectangle and to improve the convergence of the series that occur in the solution, we begin by finding the function v of the form

$$v(x, y) = ax + by + cxy + d$$

that has the same values as F at the four corners of the rectangle. The system to solve for (a, b, c, d) is

$$\begin{aligned} d &= F(0,0) \\ aL + d &= F(L,0) \\ aL + bH + cLH + d &= F(L,H) \\ bH + d &= F(0,H) \end{aligned}$$

or

$$\begin{pmatrix} 0 & 0 & 0 & 1 \\ L & 0 & 0 & 1 \\ L & H & LH & 1 \\ 0 & H & 0 & 1 \end{pmatrix} \begin{pmatrix} a \\ b \\ c \\ d \end{pmatrix} = \begin{pmatrix} F(0,0) \\ F(L,0) \\ F(L,H) \\ F(0,H) \end{pmatrix}$$

The coefficient matrix is nonsingular, so there will be a unique solution. Note that v satisfies Laplace equation

$$\frac{\partial^2 v}{\partial x^2}(x,y) + \frac{\partial^2 v}{\partial y^2}(x.y) = 0$$

for all (x,y) in the plane.

Next we find the function w such that

$$\frac{\partial^2 w}{\partial x^2}(x,y) + \frac{\partial^2 w}{\partial y^2}(x.y) = 0$$

for $0 \le x \le L$ and $0 \le y \le H$ such that

$$w(x,y) = F(x,y) - v(x,y)$$

for all (x,y) on the boundary of the rectangle. To find w we follow the procedure given in Laplace Equation Problem I to find w_1 such that

$$\begin{aligned}
\frac{\partial^2 w_1}{\partial x^2}(x,y) + \frac{\partial^2 w_1}{\partial y^2}(x.y) &= 0 \text{ for } 0 \le x \le L \text{ and } 0 \le y \le H, \\
w_1(0,y) &= 0 \text{ for } 0 \le y \le H, \\
w_1(L,y) &= 0 \text{ for } 0 \le y \le H, \\
w_1(x,H) &= 0 \text{ for } 0 \le x \le L, \text{ and} \\
w_1(x,0) &= F(x,0) - v(x,0) \text{ for } 0 \le x \le L,
\end{aligned}$$

the procedure given in Laplace Equation Problem II to find w_2 such that

$$\begin{aligned}
\frac{\partial^2 w_2}{\partial x^2}(x,y) + \frac{\partial^2 w_2}{\partial y^2}(x.y) &= 0 \text{ for } 0 \le x \le L \text{ and } 0 \le y \le H, \\
w_2(0,y) &= 0 \text{ for } 0 \le y \le H, \\
w_2(L,y) &= 0 \text{ for } 0 \le y \le H, \\
w_2(x,H) &= F(x,H) - v(x,H) \text{ for } 0 \le x \le L, \text{ and} \\
w_2(x,0) &= 0 \text{ for } 0 \le x \le L,
\end{aligned}$$

the procedure given in Laplace Equation Problem III to find w_3 such that

$$\begin{aligned}
\frac{\partial^2 w_3}{\partial x^2}(x,y) + \frac{\partial^2 w_3}{\partial y^2}(x.y) &= 0 \text{ for } 0 \le x \le L \text{ and } 0 \le y \le H, \\
w_3(0,y) &= F(0,y) - v(0,y) \text{ for } 0 \le y \le H, \\
w_3(L,y) &= 0 \text{ for } 0 \le y \le H, \\
w_3(x,H) &= 0 \text{ for } 0 \le x \le L, \text{ and} \\
w_3(x,0) &= 0 \text{ for } 0 \le x \le L,
\end{aligned}$$

and the procedure given in Laplace Equation Problem IV to find w_4 such that

$$\begin{aligned}
\frac{\partial^2 w_4}{\partial x^2}(x,y) + \frac{\partial^2 w_4}{\partial y^2}(x.y) &= 0 \text{ for } 0 \le x \le L \text{ and } 0 \le y \le H, \\
w_4(0,y) &= 0 \text{ for } 0 \le y \le H, \\
w_4(L,y) &= F(L,y) - v(L,y) \text{ for } 0 \le y \le H, \\
w_4(x,H) &= 0 \text{ for } 0 \le x \le L, \text{ and} \\
w_4(x,0) &= 0 \text{ for } 0 \le x \le L.
\end{aligned}$$

Then let

$$w = w_1 + w_2 + w_3 + w_4.$$

Finally, let

$$u(x,y) = v(x,y) + w(x,y)$$

for all (x,y) with $0 \le x \le L$ and $0 \le y \le H$.