Wave Equation Problem I

PROBLEM: Suppose that each of c and L is a positive number. Derive the solution to

$$\begin{aligned}
\frac{\partial^2 u}{\partial t^2}(x,t) &= c^2 \frac{\partial^2 u}{\partial x^2}(x,t) \text{ for } 0 \le x \le L \text{ amd all } t \text{ in } \mathbb{R}, && (1)\\
u(0,t) &= 0 \text{ for all } t \text{ in } \mathbb{R}, && (2)\\
u(L,t) &= 0 \text{ for all } t \text{ in } \mathbb{R}, && (3)\\
u(x,0) &= f(x) \text{ for } 0 \le x \le L, \text{ and} && (4)\\
\frac{\partial u}{\partial t}(x,0) &= g(x) \text{ for } 0 \le x \le L. && (5)
\end{aligned}$$

Then find the solution when

$$f(x) = x(L-x) \text{ for } 0 \le x \le L$$

and

$$g(x) = \left| x - \frac{L}{2} \right| - \frac{L}{2} \text{ for } 0 \le x \le L.$$

SOLUTION: Suppose that u is an elementary separated solution to (1). This means

$$u(x,t) = \varphi(x)h(t)$$

for some pair of one-place functions φ and h. Inserting this into (1), we have

$$\varphi(x)h''(t) = c^2\varphi''(x)h(t). \tag{6}$$

Assuming for now that

$$u(x.t) \ne 0,$$

and dividing each side of (6) by $\varphi(x)h(t)$, we have

$$\frac{\varphi(x)h''(t)}{\varphi(x)h(t)} = c^2\frac{\varphi''(x)h(t)}{\varphi(x)h(t)},$$

so

$$\frac{h''(t)}{h(t)} = c^2\frac{\varphi''(x)}{\varphi(x)}.$$

This holds for all t and all x with $0 \le x \le L$, so there is a constant K such that

$$\frac{h''(t)}{h(t)} = K = c^2\frac{\varphi''(x)}{\varphi(x)} \tag{7}$$

for all t and all x with $0 \le x \le L$. As a matter of notational convenience and so that we can more easily make use of our earlier work on two-point boundary value problems, we let

$$\lambda = -\frac{K}{c^2} \text{ so } K = -c^2\lambda.$$

From (7) we then have

$$-\varphi''(x) = \lambda\varphi(x) \text{ for all } x \text{ in } [0,L] \tag{8}$$

and

$$h''(t) = -\lambda c^2 h(t) \text{ for all } t. \tag{9}$$

It is worth noting that if

$$u(x,t) = \varphi(x)h(t)$$

and (8) and (9) hold, then

$$\begin{aligned} \frac{\partial^2 u}{\partial t^2}(x,t) &= \varphi(x)h''(t) = -\lambda c^2 \varphi(x)h(t) \\ &= c^2\varphi''(x)h(t) = c^2\frac{\partial^2 u}{\partial x^2}(x,t) \end{aligned}$$

so the PDE (1)

$$\frac{\partial^2 u}{\partial t^2}(x,t) = c^2\frac{\partial^2 u}{\partial x^2}(x,t)$$

will be satisfied, and we no longer need to assume that $u(x,t) \neq 0$. Continuing with our assumption that

$$u(x,t) = \varphi(x)h(t)$$

We have from conditions (2) and (3) (which stated that $u(0,t) = 0 = u(L,t)$) that either $h(t) = 0$ for all t which we reject because of (4) and (5) (which stated that $u(x,0) = f(x)$ and $\frac{\partial u}{\partial t}(x,0) = g(x)$) or

$$\varphi(0) = 0 \tag{10}$$

and

$$\varphi(L) = 0 \tag{11}$$

which we must accept.

The Sturm-Louville problem consisting of (8), (10), and (11) (which we repeat here)

$$\begin{aligned} -\varphi'' &= \lambda\varphi \text{ on } [0,L] \\ \varphi(0) &= 0, \text{ and} \\ \varphi(L) &= 0 \end{aligned}$$

is one which we have studied. A proper listing of eigenvalues and eigenfunctions for this problem is

$$\{\lambda_k\}_{k=1}^{\infty} \text{ and } \{\varphi_k\}_{k=1}^{\infty}$$

where

$$\lambda_k = (\frac{k\pi}{L})^2 \text{ for } k = 1, 2, \ldots$$

and

$$\varphi_k(x) = \sin\frac{k\pi}{L}x \text{ for all } x \text{ in } [0,L] \text{ and } k = 1, 2, \ldots.$$

The equation (9)

$$h''(t) = -c^2\lambda h(t)$$

is equivalent to

$$h''(t) + c^2\lambda h(t) = 0. \tag{12}$$

When $\lambda > 0$ as it must be because all eigenvalues for the problem (8), (10), and (11) are positive, a linearly independent pair of solutions to (12) is the pair whose values at t are

$$\cos\sqrt{\lambda}ct \text{ and } \sin\sqrt{\lambda}ct.$$

Thus when $\lambda = \lambda_k$ the solutions to (9) are linear combinations of the functions h_{1k} and h_{2k} where

$$h_{1k}(t) = \cos\sqrt{\lambda_k}ct \text{ and } \mathrm{h}_{2k}(t) = \sin\sqrt{\lambda_k}ct.$$

Let us recall the original problem.

$$\begin{aligned}
\frac{\partial^2 u}{\partial t^2}(x,t) &= c^2\frac{\partial^2 u}{\partial x^2}(x,t) \text{ for } 0 \le x \le L \text{ amd all } t \text{ in } \mathbb{R}, && (1)\\
u(0,t) &= 0 \text{ for all } t \text{ in } \mathbb{R}, && (2)\\
u(L,t) &= 0 \text{ for all } t \text{ in } \mathbb{R}, && (3)\\
u(x,0) &= f(x) \text{ for } 0 \le x \le L, \text{ and} && (4)\\
\frac{\partial u}{\partial t}(x,0) &= g(x) \text{ for } 0 \le x \le L. && (5)
\end{aligned}$$

The problem consisting of (1), (2), and (3) is linear and homogeneous, so if $\{A_k\}_{k=1}^n$ and $\{B_k\}_{k=1}^n$ are finite sequences of numbers and

$$u(x,t) = \sum_{k=1}^{n} \varphi_k(x)[A_k h_{1k}(t) + B_k h_{2k}(t)],$$

then u will be a solution to (1), (2), and (3). Thus we hope that the solution to the problem consisting of (1) through (5) will be of the form

$$u(x,t) = \sum_{k=1}^{\infty} \varphi_k(x)[A_k h_{1k}(t) + B_k h_{2k}(t)] \tag{13}$$

for some perhaps infinite sequences of constants $\{A_k\}_{k=1}^\infty$ and $\{B_k\}_{k=1}^\infty$.

Condition (4)

$$u(x,0) = f(x) \text{ for } x \text{ in } [0,L],$$

implies

$$f = \sum_{k=1}^{\infty} \varphi_k[A_k h_{1k}(0) + B_k h_{2k}(0)] = \sum_{k=1}^{\infty}[A_k \cos 0 + B_k \sin 0]\varphi_k = \sum_{k=1}^{\infty} A_k\varphi_k.$$

Since $\{\varphi_k\}_{k=1}^\infty$ is an orthogonal sequence of non zero functions this implies

$$A_k = \frac{< f, \varphi_k >}{< \varphi_k, \varphi_k >}$$

so for $k = 1, 2, \ldots$where the inner product is defined by

$$< \alpha, \beta >= \int_0^L \alpha(x)\beta(x)dx.$$

For this sequence $\{\varphi_k\}$,

$$< \varphi_k, \varphi_k >= \int_0^L (\sin \frac{k\pi x}{L})^2 dx = \frac{L}{2} \text{ for } k = 1, 2, \ldots.$$

Returning to (13) we expect

$$\frac{\partial u}{\partial t}(x,t) = \sum_{k=1}^{\infty} \varphi_k(x)[A_k h'_{1k}(t) + B_k h'_{2k}(t)].$$

Condition (5)

$$\frac{\partial u}{\partial t}(x, 0) = g(x) \text{ for all } x \text{ in } [0, L]$$

implies

$$g = \sum_{k=1}^{\infty} \varphi_k[A_k h'_{1k}(0) + B_k h'_{2k}(0)] = \sum_{k=1}^{\infty}(\frac{k\pi c}{L})[-A_k \sin 0 + B_k \cos 0]\varphi_k = \sum_{k=1}^{\infty}(\frac{k\pi c}{L})B_k\varphi_k$$

so

$$(\frac{k\pi c}{L})B_k = \frac{< g, \varphi_k >}{< \varphi_k, \varphi_k >} \text{ or } B_k = (\frac{2}{L})(\frac{L}{k\pi c}) < g, \varphi_k >$$

for $k = 1, 2, 3, \ldots$.In summary, the solution to the original problem (1) through (5) is u where

$$u(x,t) = \sum_{k=1}^{\infty}[A_k \cos \frac{k\pi ct}{L} + B_k \sin \frac{k\pi ct}{L}] \sin \frac{k\pi x}{L}$$

in which

$$A_k = \frac{2}{L}\int_0^L f(x) \sin \frac{k\pi x}{L} dx \text{ for } k = 1, 2, \ldots$$

and

$$B_k = \frac{2}{k\pi c}\int_0^L g(x) \sin \frac{k\pi x}{L} dx \text{ for } k = 1, 2, \ldots$$

If f is given by

$$f(x) = x(L - x) \text{ for } 0 \le x \le L$$

then

$$A_k = \frac{2}{L}\int_0^L x(L - x) \sin \frac{k\pi x}{L} dx$$

Remembering that

$$\sin k\pi = 0 \text{ and } \cos k\pi = (-1)^k$$

and integrating by parts twice, we find that

$$A_k = \frac{4L^2}{\pi^3 k^3}(1 - (-1)^k).$$

If g is given by

$$g(x) = \left| x - \frac{L}{2} \right| - \frac{L}{2} \text{ for } 0 \leq x \leq L$$

then

$$B_k = \frac{2}{k\pi c} \int_0^L \left[\left| x - \frac{L}{2} \right| - \frac{L}{2} \right] \sin \frac{k\pi x}{L} dx.$$

So

$$B_k = \frac{-2}{k\pi c} \int_0^{L/2} x \sin \frac{k\pi x}{L} dx + \frac{2}{k\pi c} \int_{L/2}^L (x - L) \sin \frac{k\pi x}{L} dx.$$

Using integration by parts, we find that

$$B_k = \frac{-4L^2 \sin \frac{1}{2} k\pi}{k^3 \pi^3 c}$$

so

$$u(x,t) = \frac{4L^2}{\pi^3} \sum_{k=1}^{\infty} \frac{1}{k^3} \left[(1 - (-1)^k) \cos \frac{k\pi ct}{L} - \frac{\sin \frac{1}{2} k\pi}{c} \sin \frac{k\pi ct}{L} \right] \sin \frac{k\pi x}{L}.$$