The Heat Equation for a Rectangular Plate

Suppose that each of κ, L, and H is a positive number. Derive the solution to the following heat equation problem.

$$\begin{aligned}
\frac{\partial u}{\partial t}(x,y,t) &= \kappa\nabla^2 u(x,y,t) \text{ for } \ 0 \le x \le L,\ 0 \le y \le H \text{ and } t \ge 0 && (1)\\
& (\nabla^2 \text{means } \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}) \\
u(0,y,t) &= 0 \text{ for } 0 \le y \le H \text{ and } t \ge 0, && (2)\\
u(L,y,t) &= 0 \text{ for } 0 \le y \le H \text{ and } t \ge 0, && (3)\\
u(x,0,t) &= 0 \text{ for } 0 \le x \le L \text{ and } t \ge 0, && (4)\\
u(x,H,t) &= 0 \text{ for } 0 \le x \le L \text{ and } t \ge 0, && (5)\\
u(x,y,0) &= F(x,y) \text{ for } 0 \le x \le L \text{ and } 0 \le y \le H. && (6)
\end{aligned}$$

Solution. Suppose that u is an elementary separated solution of the form

$$u(p,t) = \varphi(p)h(t) \tag{7}$$

for p in $[0,L] \times [0,H]$ and $t \ge 0$. Putting this into (1) we have

$$\kappa\left(\frac{\partial^2\varphi}{\partial x^2}(p) + \frac{\partial^2\varphi}{\partial y^2}(p)\right)h(t) = \varphi(p)h'(t).$$

Assuming for now that $u(p,t) \ne 0$ and dividing each side of the last equation by it we have

$$\frac{\frac{\partial^2\varphi}{\partial x^2}(p) + \frac{\partial^2\varphi}{\partial y^2}(p)}{\varphi(p)} = \frac{h'(t)}{\kappa h(t)}$$

for p in $[0,L] \times [0,H]$ and $t \ge 0$. Letting $-\lambda$ be the common constant value, we have

$$-\nabla^2\varphi(p) = \lambda\varphi(p) \text{ for } p \text{ in } [0,L] \times [0,H] \tag{8}$$

and

$$h'(t) + \lambda\kappa h(t) = 0 \text{ for } t \ge 0. \tag{9}$$

If (7), (8), and (9) hold then (1) holds and we no longer need to assume that $u(p,t) \ne 0$. The boundary conditions (2)-(5) and the fact that u cannot be the zero function because of (6) imply

$$\begin{aligned}
\varphi(0,y) &= 0 \text{ for } 0 \le y \le H, && (10)\\
\varphi(L,y) &= 0 \text{ for } 0 \le y \le H, && (11)\\
\varphi(x,0) &= 0 \text{ for } 0 \le x \le L, \text{and} && (12)\\
\varphi(x,H) &= 0 \text{ for } 0 \le x \le L. && (13)
\end{aligned}$$

A proper listing of eigenvalues and eigenfunctions for (8) and (10)-(13) is

$$\{\lambda_{kj}\}_{k=1,j=1}^{\infty} \text{ and } \{\varphi_{kj}\}_{k=1,j=1}^{\infty}$$

where $\lambda_{kj} = (\frac{k\pi}{L})^2 + (\frac{j\pi}{H})^2$ and $\varphi_{kj}(x,y) = \sin\frac{k\pi x}{L}\sin\frac{j\pi y}{H}$

The solutions to (9) are multiples of h_{kj} where

$$h_{kj}(z) = \exp(-\kappa\lambda_{kj}t)$$

Anything of the form

$$\sum_{k=1}^{m}\sum_{j=1}^{n} E_{kj}\varphi_{kj}(x,y)h_{kj}(t)$$

will be a solution to the homogeneous problem (1)-(6), but to also get (7) we expect that

$$u(x,y,t) = \sum_{k=1}^{\infty}\sum_{j=1}^{\infty} E_{kj}\varphi_{kj}(x,y)h_{kj}(t).$$

The coefficients are determined by (6).

$$F(x,y) = \sum_{k=1}^{\infty}\sum_{j=1}^{\infty} E_{kj}\varphi_{kj}(x,y)h_{kj}(0).$$

Note that

$$h_{kj}(0) = 1$$

so

$$E_{kj} = \frac{< F, \varphi_{kj} >}{< \varphi_{kj}, \varphi_{kj} >}.$$

Thus

$$u(x,y,z) = \sum_{k=1}^{\infty}\sum_{j=1}^{\infty} E_{kj}\sin\frac{k\pi x}{L}\sin\frac{j\pi y}{H}\exp(-\kappa\lambda_{kj}t)$$

where E_{kj} is as above and

$$\lambda_{kj} = (\frac{k\pi}{L})^2 + (\frac{j\pi}{H})^2.$$