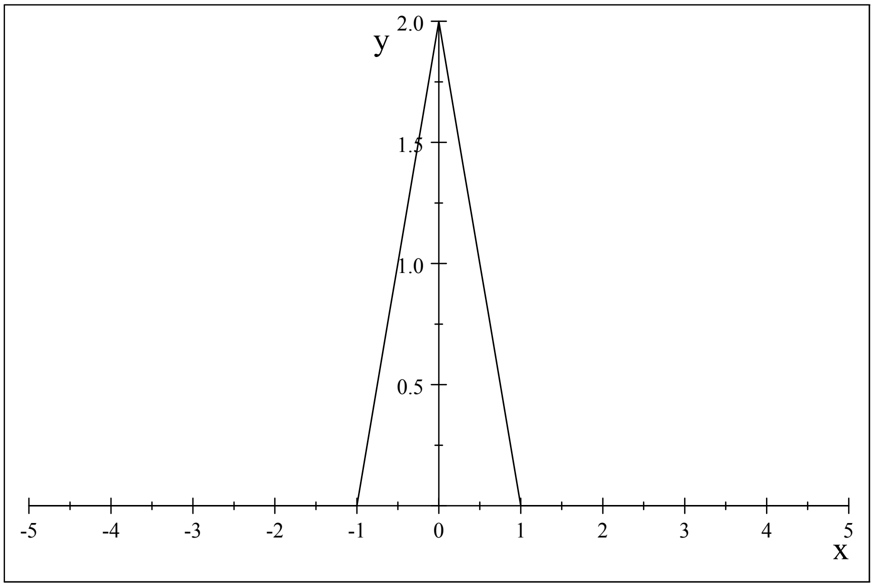
Snapshots of the String

1. Let

$$\varphi(x) = \begin{cases} 0 & if \quad x < -1 \\ 2x+2 & if \quad -1 \le x < 0 \\ -2x+2 & if \quad 0 \le x < 1 \\ 0 & if \quad 1 \le x \end{cases}$$



Graph of φ

Let

$$\psi(x) = 0 \text{ for all } x,$$

and let u be the solution to the wave equation

$$\frac{\partial^2 u}{\partial t^2}(x,t) = \frac{\partial^2 u}{\partial x^2}(x,t)$$

for all (x,t) such that

$$\begin{aligned} u(x,0) &= \varphi(x), \text{ and} \\ \frac{\partial u}{\partial t}(x,0) &= \psi(x) \text{ for all } x. \end{aligned}$$

Let

$$h(x) = u(x, t_0) \text{ for all } x.$$

Draw the graph of h over the interval $[-4,4]$ when

(a) $t_0 = 0.10$,

(b) $t_0 = 0.40$,

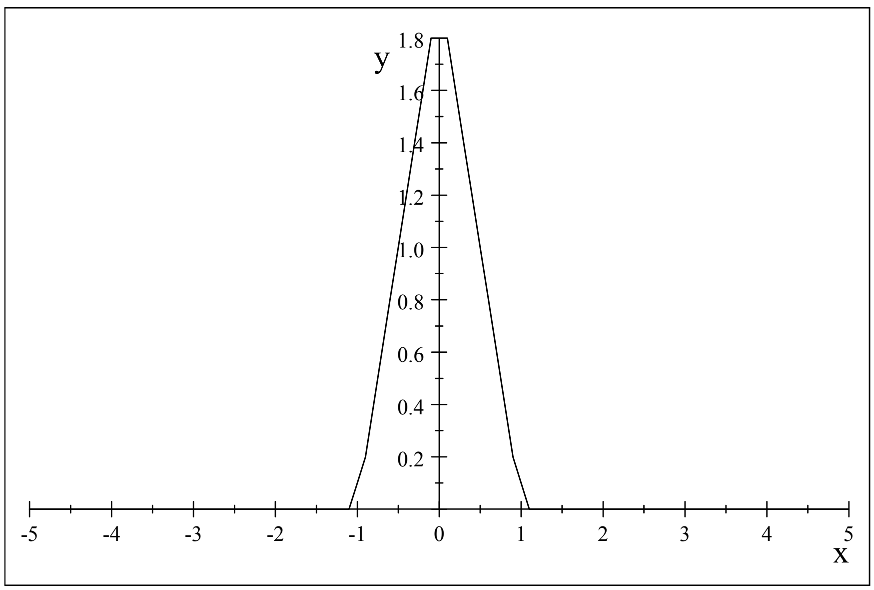
(c) $t_0 = 0.80$, and

(d) $t_0 = 1.20$

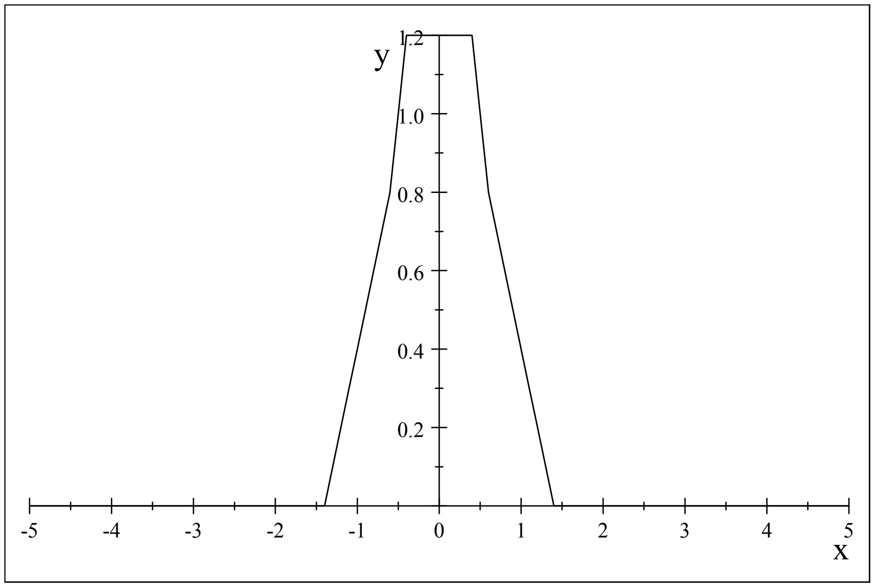
Solution. Using d'Alembert's solution

$$h(x) = \frac{1}{2}\varphi(x + t_0) + \frac{1}{2}\varphi(x - t_0)$$

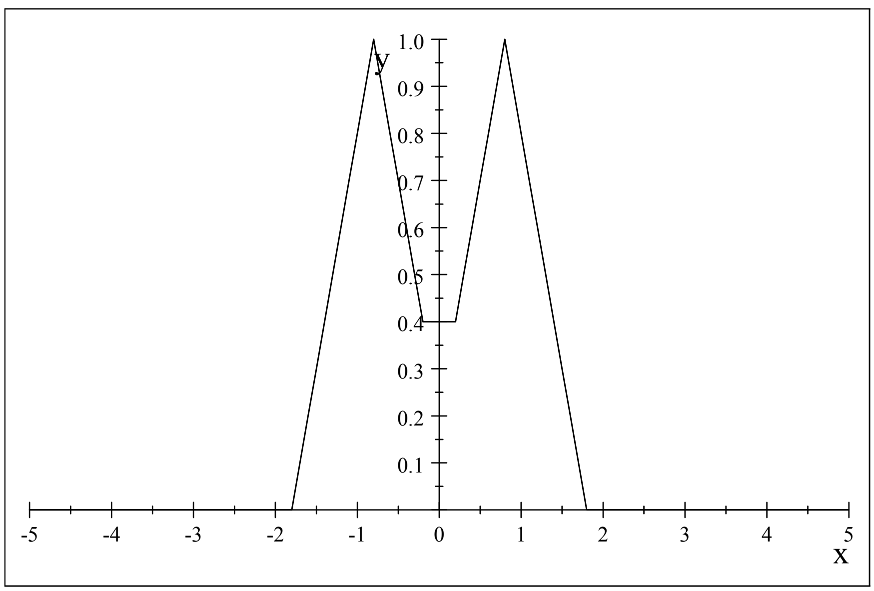
so the graph of h is obtained by adding the graph of $\frac{1}{2}\varphi$ shifted t_0 units to the left and the graph of $\frac{1}{2}\varphi$ shifted t_0 units to the right.



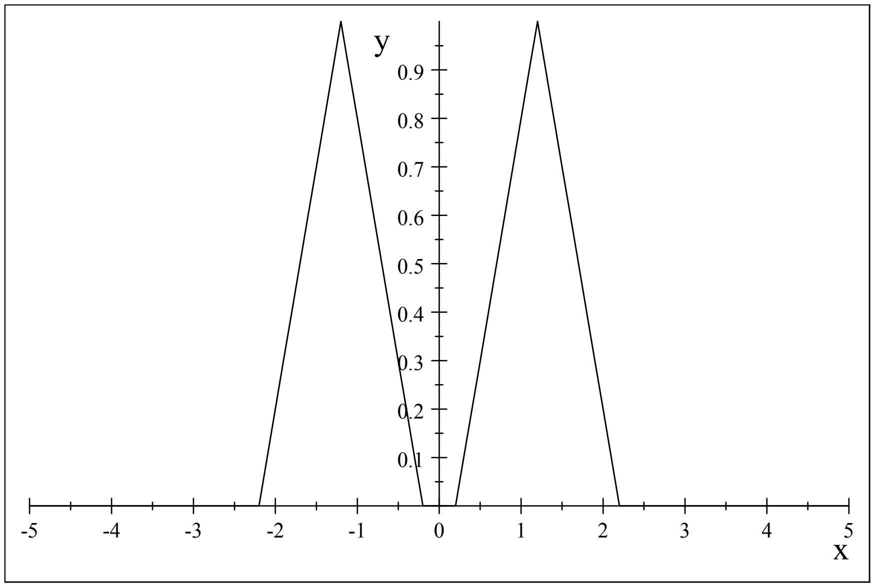
Graph of h when $t_0 = 0.1$



Graph of h when $t_0 = 0.4$



Graph of h when $t_0 = 0.8$



Graph of h when $t_0 = 1.2$

2. Let u be the solution to

$$\begin{aligned}\frac{\partial^2 u}{\partial t^2}(x,t) &= \frac{\partial^2 u}{\partial x^2}(x,t) \text{ for all } x \text{ and } t \text{ in } \mathbb{R},\\ u(x,0) &= 0 \text{ for all } x \text{ in } \mathbb{R}, \text{ and}\\ \frac{\partial u}{\partial t}(x,0) &= \psi(x) \text{ for all } x \text{ in } \mathbb{R}.\end{aligned}$$

where

$$\psi(x) = \begin{cases} 0 & \text{for} \quad x < -2 \\ 2 & \text{for} \quad -2 < x < -1 \\ -2 & \text{for} \quad -1 < x < 0 \\ 0 & \text{for} \quad x > 0 \end{cases}.$$

Let

$$h(x) = u(x, t_0) \text{ for all } x \text{ in } \mathbb{R}.$$

Sketch the graph of h when $t_0 = 0.1$, 0.4, 0.8, 1.0, and 3.0.

Solution. The solution u to the wave equation problem is given by

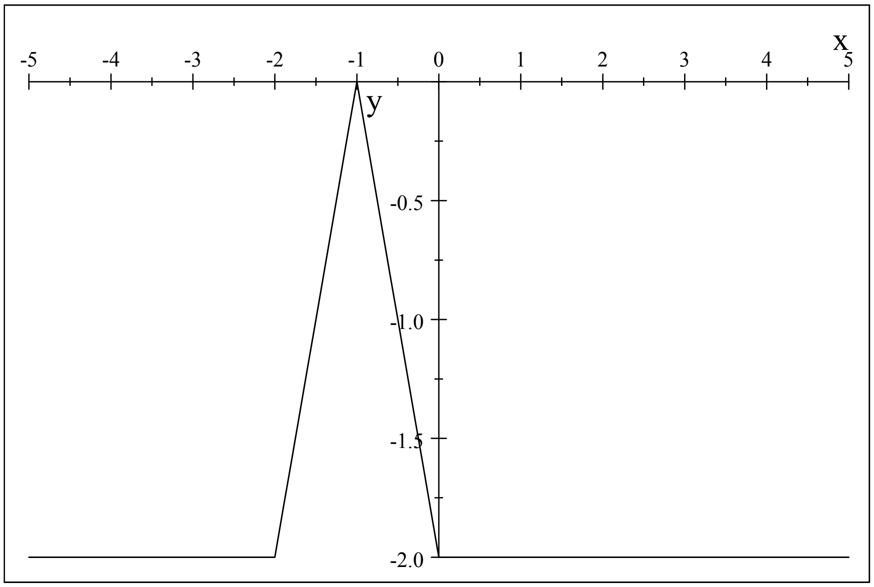
$$u(x,t) = \frac{1}{2}\int_{x-t}^{x+t} \psi(s)ds = \frac{1}{2}F(x+t) - \frac{1}{2}F(x-t)$$

where

$$F(x) = \int_0^x \psi(s)ds.$$

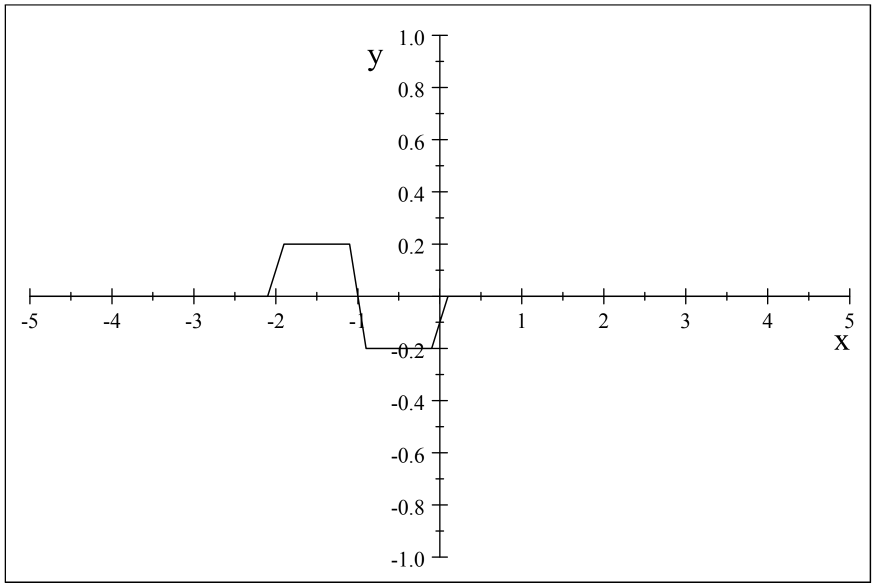
F is continuous, $F(0) = 0$, and $F' = \psi$ where ψ is defined , so F is as follows.

$$F(x) = \begin{cases} -2 & if \quad x < -2 \\ 2x+2 & if \quad -2 \le x < -1 \\ -2x-2 & if \quad -1 \le x < 0 \\ -2 & if \quad 0 \le x \end{cases}$$

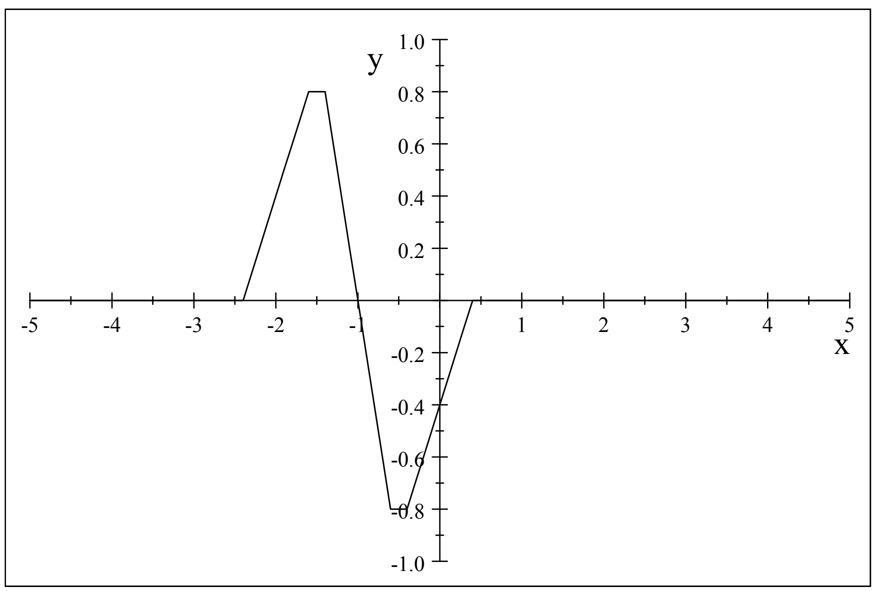


Graph of F

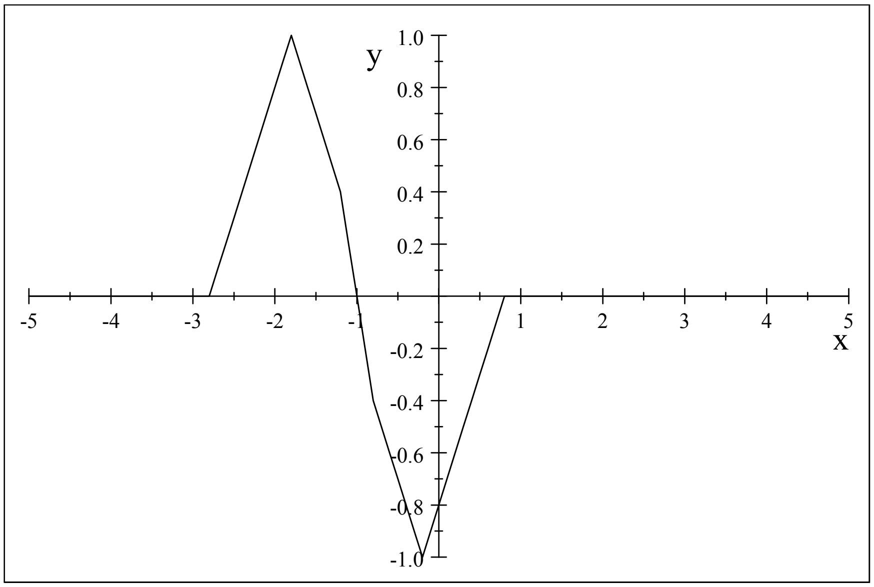
The graph of h is obtained by adding the graph of $\frac{1}{2}F$ shifted t_0 units to the left and the graph of $-\frac{1}{2}F$ shifted three units to the right.



Graph of h when $t_0 = 0.1$

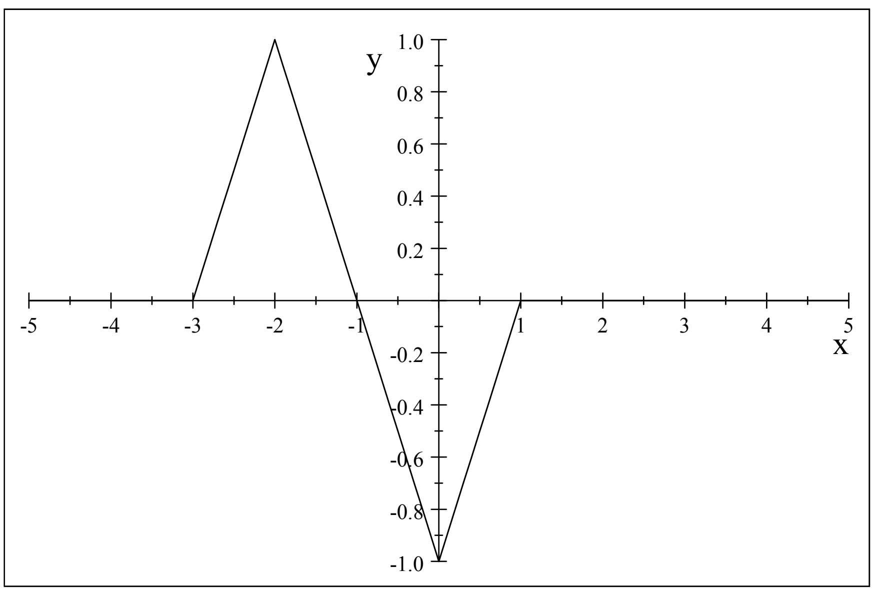


Graph of h when $t_0 = 0.4$

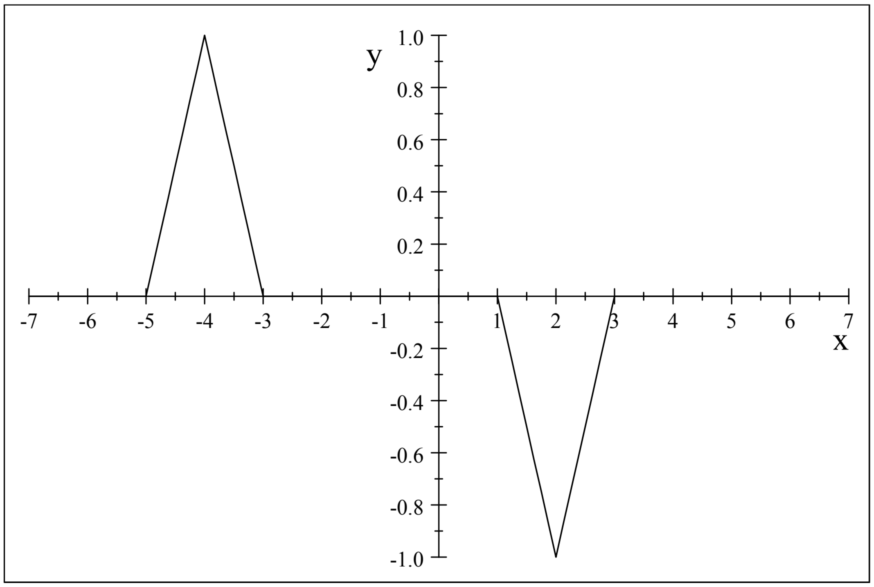


Graph of h when $t_0 = 0.8$

$$\frac{1}{2}F(x+1.0) - \frac{1}{2}F(x-1.0)$$



Graph of h when $t_0 = 1.0$



Graph of h when $t_0 = 3.0$

3. Let

$$\varphi(x) = \begin{cases} 0 & if \quad x < -1 \\ -2x-2 & if \quad -1 \le x < 0 \\ 2x-2 & if \quad 0 \le x < 1 \\ 0 & if \quad 1 \le x \end{cases},$$

$$\psi(x) = \begin{cases} 0 & if \quad x < -1 \\ 1 & if \quad -1 < x < 0 \\ -1 & if \quad 0 < x < 1 \\ 0 & if \quad 1 < x \end{cases},$$

and let u be the solution to the wave equation

$$\frac{\partial^2 u}{\partial t^2}(x,t) = \frac{\partial^2 u}{\partial x^2}(x,t)$$

for all (x,t) such that

$$\begin{aligned} u(x,0) &= \varphi(x), \text{ and} \\ \frac{\partial u}{\partial t}(x,0) &= \psi(x) \text{ for all } x. \end{aligned}$$

Let

$$h(x) = u(x,4) \text{ for all } x.$$

Draw the graph of h over the interval $[-10, 10]$.

Solution.

$$u(x,t) = \frac{1}{2}(\varphi(x+t) + \varphi(x-t)) + \frac{1}{2}F(x+t) - \frac{1}{2}F(x-t)$$

where

$$F(x)=\int_0^x \psi(s)ds$$

so

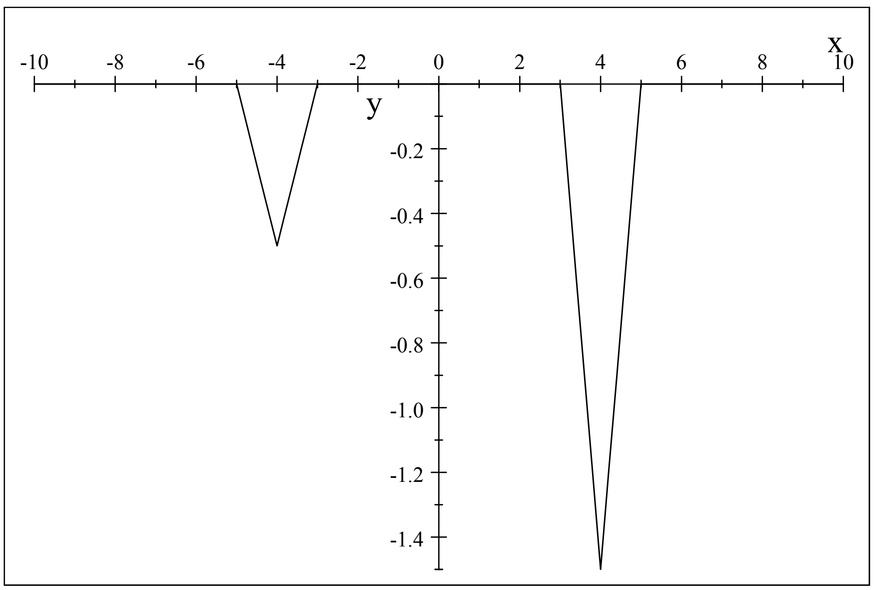
$$F(x)=\begin{cases} -1 & if \quad x\le -1 \\ x & if \quad -1<x\le 0 \\ -x & if \quad 0<x\le 1 \\ -1 & if \quad 1<x \end{cases}$$

$$h(x)=\frac{1}{2}(\varphi(x+4)+\varphi(x-4))+\frac{1}{2}F(x+4)-\frac{1}{2}F(x-4)$$

$$h_1(x)=\frac{1}{2}(\varphi(x+4)+\varphi(x-4))$$

$$h_2(x)=\frac{1}{2}F(x+4)-\frac{1}{2}F(x-4)$$

$$h(x)=h_1(x)+h_2(x)$$



h

4. Let

$$f(x)=\begin{cases} 0 & if \quad 0\le x<1 \\ 2x-2 & if \quad 1\le x<2 \\ -2x+6 & if \quad 2\le x<3 \\ 0 & if \quad 3\le x\le 10 \end{cases},$$

$$g(x)=0 \text{ for } 0\le x\le 10,$$

and let u be the solution to the wave equation

$$\frac{\partial^2 u}{\partial t^2}(x,t) = \frac{\partial^2 u}{\partial x^2}(x,t)$$

for $0 \leq x \leq 10$ and all t such that

$$\begin{aligned} u(0,t) &= 0 \text{ for all } t, \\ u(10,t) &= 0 \text{ for all } t, \\ u(x,0) &= f(x) \text{ for } 0 \leq x \leq 10, \text{and} \\ \frac{\partial u}{\partial t}(x,0) &= g(x) \text{ for } 0 \leq x \leq 10. \end{aligned}$$

Let

$$h(x) = u(x,t_0) \text{ for } 0 \leq x \leq 10.$$

Draw the graph of h over the interval $[0,10]$ when

(a) $t_0 = 4$,

(b) $t_0 = 6$,

(c) $t_0 = 8$, and

(d) $t_0 = 12$.

To do this, let φ be the period 20 extension of the odd $[-10,10]$ extension of f and let $\psi(x) = 0$ for all x. Let u be the d'Alembert solution determined by φ and ψ with $c = 1$. Then h is also given by

$$h(x) = u(x,t_0) \text{ for } 0 \leq x \leq 10$$

where u is the d'Alembert solution.

Solution. Let

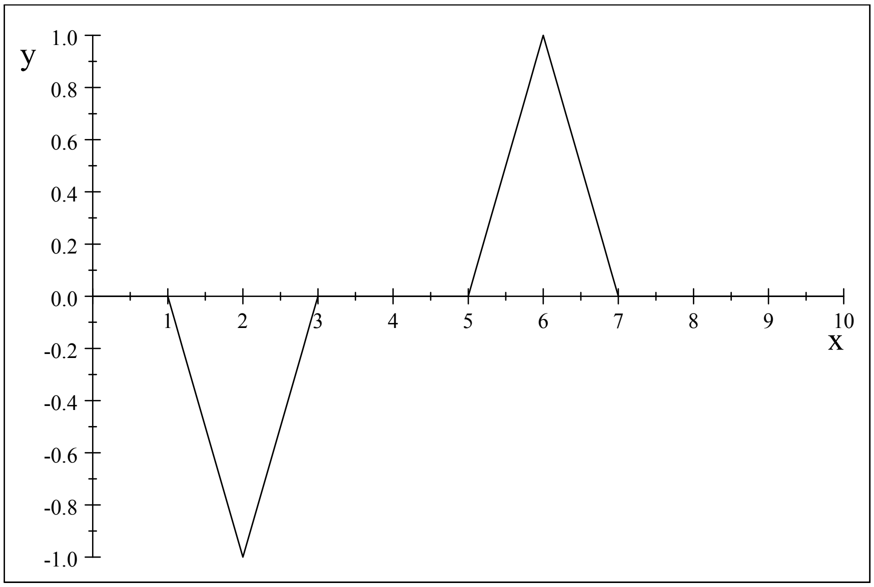
$$\alpha(x) = \begin{cases} -f(-x) & if \quad -10 \leq x < 0 \\ f(x) & if \quad 0 \leq x \leq 10 \end{cases}$$

and let

$$\varphi_2(x) = \begin{cases} \alpha(x+20) & if \quad -30 \leq x < -10 \\ \alpha(x) & if \quad -10 \leq x \leq 10 \\ \alpha(x-20) & if \quad 10 \leq x \leq 30 \end{cases}$$

Part a.

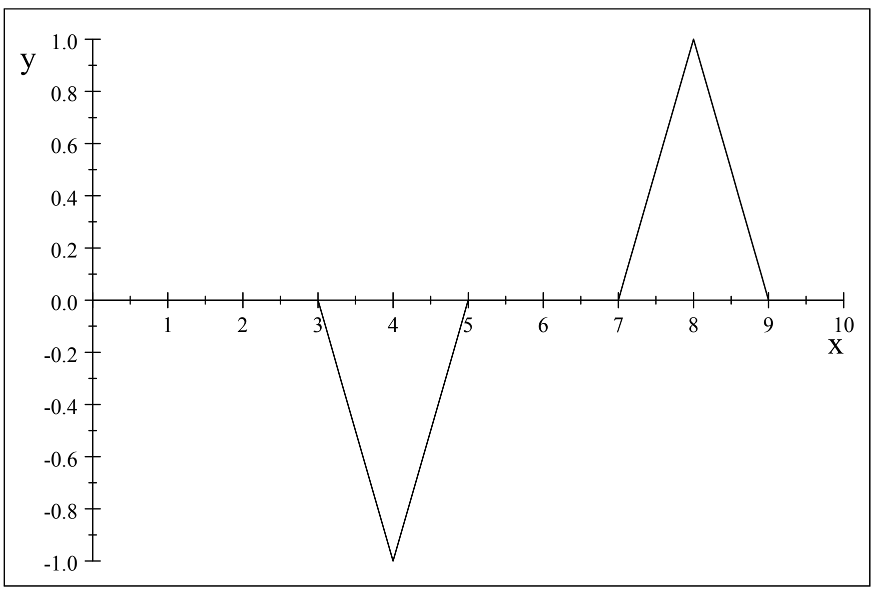
$$h(x) = \frac{1}{2}(\varphi_2(x+4) + \varphi_2(x-4))$$



$t_0 = 4$

Part b.

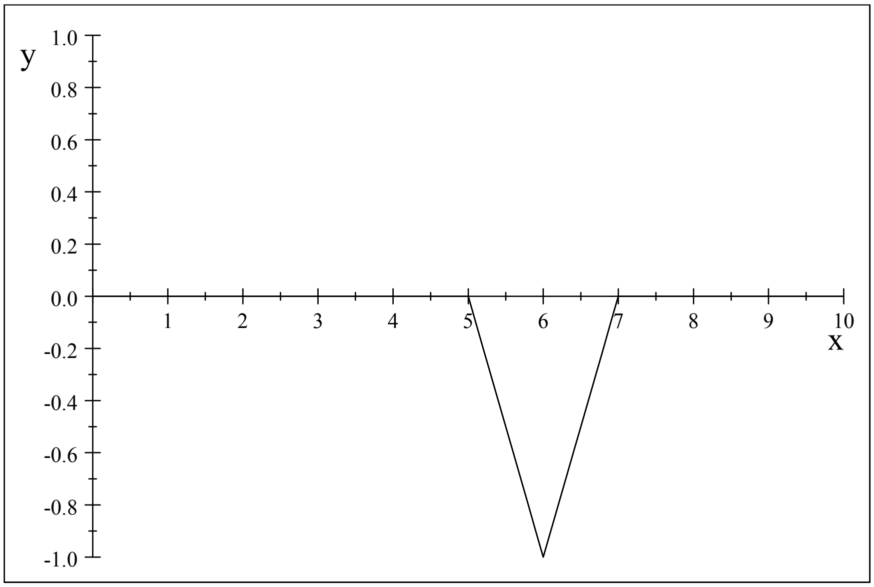
$$h(x) = \frac{1}{2}(\varphi_2(x+6) + \varphi_2(x-6))$$



$t_0 = 6$

Part c.

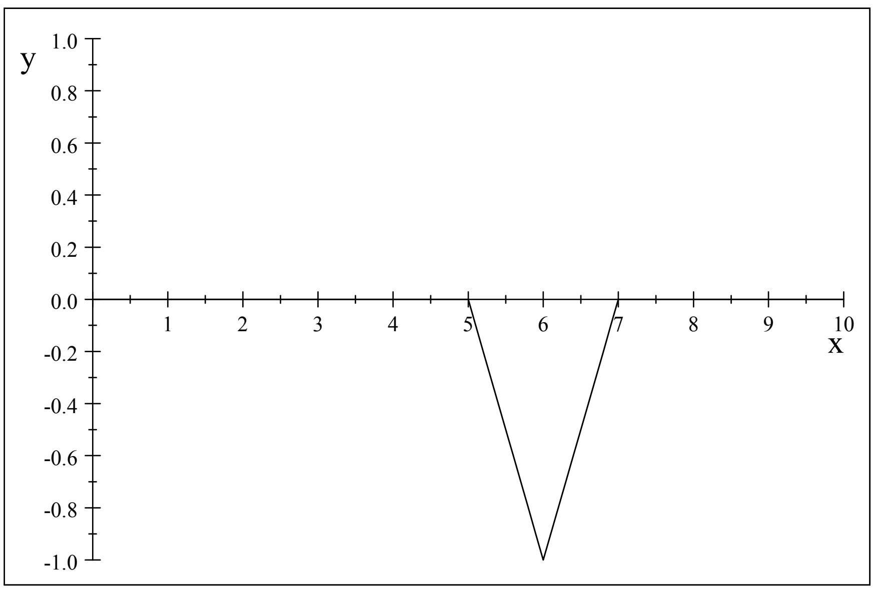
$$h(x) = \frac{1}{2}(\varphi_2(x+8) + \varphi_2(x-8))$$



$t_0 = 8$

Part d.

$$h(x) = \frac{1}{2}(\varphi_2(x+12) + \varphi_2(x-12))$$



$t_0 = 12$