Department of Mathematics
University of Houston
Numerical Analysis I
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Numerical Mathematics I
(1. Homework Assignment)

Exercise 1 (*Gauss elimination for diagonally dominant matrices*)

A matrix $A \in \mathbf{R}^{n\times n}$ is called diagonally dominant, if

$$| a_{ii} | \geq \sum_{j\neq i} | a_{ij} | \quad , \quad 1 \leq i \leq n \ .$$

Let $A^{(k)}$, $0 \leq k \leq n-1$, with $A^{(0)} := A$ be the intermediate matrices that arise during Gauss elimination.
Show by an induction argument that the matrices $A^{(k)}$, $1 \leq k \leq n-1$, are also diagonally dominant and satisfy

$$\sum_{j=k+1}^{n} | a_{ij}^{(k)} | \leq \sum_{j=1}^{n} | a_{ij} | \ .$$

Show further that this implies

$$| a_{ij}^{(k)} | \leq 2 \max_{i,j} | a_{ij} | \ .$$

Exercise 2 (*Avoiding fill-In*)

Let $A \in \mathbf{R}^{5\times 5}$ be a regular matrix of the following form

$$A = \begin{pmatrix} \times & \times & \times & \times & \times \\ \times & \times & & & \\ \times & & \times & & \\ \times & & & \times & \\ \times & & & & \times \end{pmatrix} \ .$$

Here, the symbol $\times$ denotes a nonzero entry, whereas all other matrix entries are supposed to be zero.

(i) Show that in case of Gauss elimination without pivoting, the first step already generates nonzero entries at all places where the original entry was zero ("fill-in"). Why is such a fill-in unwanted from an algorithmic point of view?

(ii) Compute permutation matrices P_1 and P_2 such that the application of Gauss elimination to $P_1 A P_2$ does not lead to fill-in.

Exercise 3 (*LR-decomposition*)

Let $A \in \mathbf{R}^{(n+1)\times(n+1)}$ be a matrix of the form

$$A \;=\; \begin{pmatrix} R & v \\ u^T & 0 \end{pmatrix} \;,$$

where $R \in \mathbf{R}^{n\times n}$ is a regular upper triangular matrix and $u, v \in \mathbf{R}^n$.

(i) Compute the LR-decomposition of A.

(ii) Show that A is regular if and only if $u^T R^{-1} v \neq 0$.

Exercise 4 (*Sherman-Morrison-Woodbury formula*)

Let $A \in \mathbf{R}^{n\times n}$, $u, v \in \mathbf{R}^n$ and assume that

$$\bar{A} \;:=\; A \,+\, u\, v^T$$

is a rank 1 update of A.

(i) Show that the matrix $\bar{A}$ is regular if and only if

$$1 \,+\, v^T A^{-1} u \;\neq\; 0 \quad .$$

(ii) In case that $\bar{A}$ is regular, prove the so-called Sherman-Morrison-Woodbury formula

$$\bar{A}^{-1} \;=\; A^{-1} \,-\, \tau^{-1} A^{-1} u\, v^T A^{-1} \quad , \quad \tau \;:=\; 1 \,+\, v^T A^{-1} u \;.$$

Delivery of the homework at latest on September 8, 2005. The homework may be submitted either electronically (rohop@math.uh.edu) or as a hardcopy in class.