



## Numerical Analysis I (2nd Homework Assignment)

### Exercise 5 (*Block Gauss elimination*)

Let  $A \in \mathbf{R}^{N \times N}$ ,  $N := \sum_{i=1}^m n_i$ ,  $n_i \in \mathbf{N}$ ,  $1 \leq i \leq m$  and  $b \in \mathbf{R}^N$  be block structured according to

$$A = \begin{pmatrix} A_{11} & \cdot & \cdot & \cdot & A_{1m} \\ A_{21} & \cdot & \cdot & \cdot & A_{2m} \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ A_{m1} & \cdot & \cdot & \cdot & A_{mm} \end{pmatrix}, \quad b = (b_1, \cdot, \cdot, \cdot, b_m)^T,$$

where  $A_{ij} \in \mathbf{R}^{n_i \times n_j}$ ,  $1 \leq i, j \leq m$ ,  $b_i \in \mathbf{R}^{n_i}$ ,  $1 \leq i \leq m$ .

(i) Consider the solution of the linear algebraic system

$$Ax = b.$$

Use a corresponding structuring of the solution vector  $x \in \mathbf{R}^N$  and give a block variant of Gauss elimination.

(ii) Give a block variant of the LR-decomposition for block tridiagonal matrices  $A \in \mathbf{R}^{N \times N}$ , i.e.,  $A_{ij} = 0$  for  $|i - j| \geq 2$ ,  $1 \leq i, j \leq m$ .

### Exercise 6 (*Schur complement*)

Let  $A \in \mathbf{R}^{N \times N}$ ,  $N := n_1 + n_2$ ,  $n_i \in \mathbf{N}$ ,  $1 \leq i \leq 2$  be a symmetric positive definite block matrix of the form

$$A = \begin{pmatrix} A_{11} & A_{12} \\ A_{12}^T & A_{22} \end{pmatrix}.$$

Prove that the Schur complement  $S = A_{22} - A_{12}^T A_{11}^{-1} A_{12}$  is also symmetric positive definite.

### Exercise 7 (*Positive definite symmetric Toeplitz matrices*)

A matrix  $A \in \mathbf{R}^{n \times n}$  is called a normalized symmetric Toeplitz matrix, if

$$a_{ij} = r_{|i-j|}, \quad 1 \leq i, j \leq n,$$

where  $r_0 = 1$  and  $r = (r_1, \dots, r_n)^T \in \mathbf{R}^n$  such that  $A$  is positive definite. The solution of the linear algebraic system

$$Ax = -r$$

is referred to as the Yule-Walker problem which plays a significant role in algorithms for the reconstruction of noisy signals.

Consider a partitioning of the matrix  $A$  according to

$$A = \begin{pmatrix} \tilde{A} & P_{n-1} \tilde{r} \\ \tilde{r}^T P_{n-1} & 1 \end{pmatrix},$$

where  $\tilde{A} \in \mathbf{R}^{(n-1) \times (n-1)}$ ,  $\tilde{r} = (r_1, \dots, r_{n-1})^T$  and

$$P_{n-1} = \begin{pmatrix} 0 & 0 & 0 & \cdot & \cdot & 0 & 0 & 1 \\ 0 & 0 & 0 & \cdot & \cdot & 0 & 1 & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ 0 & 1 & 0 & \cdot & \cdot & 0 & 0 & 0 \\ 1 & 0 & 0 & \cdot & \cdot & 0 & 0 & 0 \end{pmatrix}.$$

Use a corresponding partitioning of the right-hand side and of the solution vector. Develop a recursive algorithm for the Yule-Walker problem which computes the solution for the dimension  $n$ , provided the solution for the dimension  $n - 1$  is known.

**Exercise 8** (*Cholesky decomposition of positive semidefinite matrices*)

Assume that  $A \in \mathbf{R}^{n \times n}$  is positive semidefinite of rank  $r < n$ . Prove the following two assertions:

(i) There exists an upper triangular matrix  $R$  with nonnegative diagonal elements such that

$$A = R^T R.$$

(ii) There exists a permutation matrix  $P$  such that  $P^T A P$  has a unique Cholesky decomposition of the form

$$P^T A P = R^T R, \quad R = \begin{pmatrix} R_{11} & R_{12} \\ 0 & 0 \end{pmatrix},$$

where  $R_{11}$  is an  $r \times r$  upper triangular matrix with positive diagonal elements.

**Delivery of the homework at latest on September 22, 2005. The homework may be submitted either electronically (rohop@math.uh.edu) or as a hardcopy in class.**