Department of Mathematics
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Numerical Analysis I
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Numerical Analysis I
(5th Homework Assignment)

Exercise 17 (*Reformulation of linear least squares problems*)

Let $A \in \mathbf{R}^{m\times n}$, $m > n$, $\operatorname{rank} A = n$, $b \in \mathbf{R}^m$. The linear least squares problem

$$(*) \qquad \| Ax - b \|_2 = \min$$

can be formulated as the linear algebraic system

$$(**) \qquad \begin{pmatrix} I_m & A \\ A^T & 0 \end{pmatrix} \begin{pmatrix} r \\ x \end{pmatrix} = \begin{pmatrix} b \\ 0 \end{pmatrix} ,$$

where I_m stands for the $m \times m$ unit matrix and $r := b - Ax \in \mathbf{R}^m$ is the residual.

(i) Using the normal equations, show that the component x of the solution of $(**)$ solves the linear least squares prolem $(*)$.

(ii) Given a decomposition of A according to

$$(\dagger) \qquad A = Q \begin{pmatrix} R \\ 0 \end{pmatrix}$$

with an orthogonal matrix $Q \in \mathbf{R}^{m\times m}$ and a regular upper triangular matrix $R \in \mathbf{R}^{n\times n}$ show that by orthogonal row and column transformations the linear system

$$\begin{pmatrix} I_m & A \\ A^T & 0 \end{pmatrix} \begin{pmatrix} p \\ z \end{pmatrix} = \begin{pmatrix} f \\ g \end{pmatrix}$$

can be transformed to the form

$$\begin{pmatrix} I_n & 0 & R \\ 0 & I_{m-n} & 0 \\ R^T & 0 & 0 \end{pmatrix} \begin{pmatrix} h \\ d \\ z \end{pmatrix} = \begin{pmatrix} f_1 \\ d \\ g \end{pmatrix} .$$

Specify the relation between the vectors h, d, f_1 and f, p.

Exercise 18 (*Least squares polynomial approximation*)

Let $f : [0.1] \to \mathbf{R}$ be a continuous function. We are looking for a polynomial

$$P^{(m)}(t) \ := \ \sum_{k=0}^{n-1} x_k^{(m)} t^k \ ,$$

such that for $t_\ell := \frac{\ell}{m}$, $m > n$, there holds:

$$\frac{1}{m} \sum_{\ell=0}^{m} (f(t_\ell) \ - \ P^{(m)}(t_\ell))^2 \ = \ \min \ .$$

(i) Formulate the problem as a linear least squares problem

$$\| A^{(m)} x^{(m)} \ - \ b^{(m)} \|_2 \ = \ \min$$

with respect to the coefficients $x^{(m)} = (x_0^{(m)}, ..., x_{n-1}^{(m)})$. In particular, specify $A^{(m)}$ and $b^{(m)}$.

(ii) Determine a linear system such that its solution is the limit of the sequence $x^{(m)}$ für $m \to \infty$.

Exercise 19 (*Updating least squares problems*)

Let $A \in \mathbf{R}^{m \times n}$, $m > n$, $\operatorname{rank} A = n$ and $u \in \mathbf{R}^m$, $v \in \mathbf{R}^n$. Consider

$$\hat{A} \ := \ A \ + \ u v^T$$

and show: If $\operatorname{rank} \hat{A} = n$ and if

$$A \ = \ Q \begin{pmatrix} R \\ 0 \end{pmatrix}$$

is the QR-decomposition of A, then the QR-decomposition of $\hat{A}$ is given by

$$\hat{A} \ = \ \hat{Q} \begin{pmatrix} \hat{R} \\ 0 \end{pmatrix} \ ,$$

where

$$\hat{Q} \ = \ QU \quad , \quad U \ = \ F_{m-1} ... F_1 \hat{F}_1 ... \hat{F}_n$$

with suitable Given rotations F_k, $1 \leq k \leq m-1$, $\hat{F}_\ell$, $1 \leq \ell \leq n$.

[Hint: First, compute F_k, $1 \leq k \leq m-1$ such that

$$F_1^T ... F_{m-1}^T Q^T u \ = \ \alpha e_1 \quad , \quad \alpha \ = \ \pm \|Q^T u\|_2 \ .$$

Show that

$$\hat{H} \ = \ \hat{F}_1^T ... \hat{F}_n^T \left[\begin{pmatrix} R \\ 0 \end{pmatrix} \ + \ (Q^T u) v^T \right]$$

is an upper Hessenberg matrix. Then, choose $\hat{F}_\ell$, $1 \leq \ell \leq n$, dementsprechend.]

Exercise 20 (*Downdating least squares problems*)

For linear least squares problems, 'downdating' refers to the effect of eliminating an observation:
Let $A \in \mathbf{R}^{m \times n}$, $m \geq n$, mit rank $A = n$ and let

$$A \; = \; Q \begin{pmatrix} R \\ 0 \end{pmatrix}$$

be the QR-decomposition of A. Partition A according to

$$A \; = \; \begin{pmatrix} a_1^T \\ \tilde{A} \end{pmatrix} \quad ,$$

where $\tilde{A} \in \mathbf{R}^{(m-1) \times n}$ and a_1^T is the forst row of A and show:
The QR-decomposition of $\tilde{A}$ is given by

$$\tilde{A} \; = \; \tilde{Q} \begin{pmatrix} \tilde{R} \\ 0 \end{pmatrix} \quad , \quad \bar{Q} \; = QF_{m-1}...F_1 \; = \; \begin{pmatrix} \alpha & 0 \\ 0 & \tilde{Q} \end{pmatrix} \quad , \; | \, \alpha \, | = 1$$

with suitably chosen Givens rotations F_k , $1 \leq k \leq m-1$.

[Hint: Either apply Exercise 19 or show that the 'downdating' problem is equivalent to the QR-decomposition of the extended matrix $(e_1, A) \in \mathbf{R}^{m \times (n+1)}$.]

Delivery of the homework at latest on October 13, 2005. The homework may be submitted either electronically (rohop@math.uh.edu) or as a hardcopy in class.