Department of Mathematics
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Numerical Analysis I
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Numerical Analysis I
(2nd Practical Homework Assignment)

Practical Exdercise 2 (*cg method*)

The deflection $u = u(x)$, $x = (x_1, x_2)^T \in \Omega := (0,1)^2$ of a clamped membrane due to an exterior force of force density $f = f(x)$, $x \in \Omega$ can be described by the boundary value problem

$$\begin{aligned} -\Delta u(x) \;=\; -\,(\frac{\partial}{\partial x_1^2}u(x) \;+\; \frac{\partial}{\partial x_2^2}u(x)) \;&=\; f(x) \quad , \quad x \in \Omega \\ u(x) \;&=\; g(x) \quad , \quad x \in \Gamma := \partial\Omega \;. \end{aligned}$$

The approximation of the 2D Laplacian $-\Delta := \frac{\partial}{\partial x_1^2} + \frac{\partial}{\partial x_2^2}$ by finite differences

$$\begin{aligned} \frac{\partial}{\partial x_1^2}\,u(x) \;&\sim\; \frac{u(x_1 - h, x_2) \;-\; 2\,u(x_1, x_2) \;+\; u(x_1 + h, x_2)}{h^2} \;, \\ \frac{\partial}{\partial x_2^2}\,u(x) \;&\sim\; \frac{u(x_1, x_2 - h) \;-\; 2\,u(x_1, x_2) \;+\; u(x_1, x_2 + h)}{h^2} \end{aligned}$$

with respect to a uniform grid of mesh size $h > 0$ results in a linear algebraic system with the coefficient matrix $A = (a_{ij})_{i,j=0}^{n-1} \in \mathbb{R}^{n \times n}, n = k^2$, $k \in \mathbb{N}$, which is given by

$$a_{ij} \;=\; \begin{cases} \;\;4 & ; \quad i = j \\ -1 & ; \quad (iDIVk) = (jDIVk) \text{ and } |i-j| = 1 \\ -1 & ; \quad |i-j| = k \\ \;\;0 & ; \quad \text{sonst} \end{cases} \quad .$$

Here, $iDIVk$ stands for the integer part of i/k.

Write a code which realizes the preconditioned cg method applied to the linear system $Ax = b$ with the symmetric positive definite matrix A as given above.

As a termination criterion for the iteration use

$$\frac{\|r_i\|_2^2}{\|r_0\|_2^2} \leq \varepsilon ,$$

where r_i is the residual with respect to the i-th iteration and $\|\cdot\|_2$ refers to the Euclidean norm. During the iteration, compute

$$\rho_i := \left(\frac{\|r_i\|_2}{\|r_1\|_2} \right)^{1/(i-1)} , \ i \geq 2$$

as an estimate of the asymptotic convergence rate.

As the right-hand side $b = (b_0, ..., b_{n-1})^T$ choose $b_i = 1 , \ 0 \leq i \leq n-1$ and use the zero vector as the initial iterate. Further, choose $\varepsilon = 10^{-7}$ as the tolerance in the termination criterion.
For $k \in \{10, 20, 30, ..., 100\}$, compare the estimated asymptotic convergence rates and the total number of iterations in case of the

- cg method without preconditioning,
- cg method with the symmetric Gauss Seidel iteration as a preconditioner (SGS-pcg).

Delivery of an executable program at latest on October 27, 2005. Submit a program listing and the following table containing the total number of iterations and computed asymptotic convergence rates

	cg		**SGS-pcg**	
k	No. It.	ρ	No. It.	ρ
10				
20				
30				
.				
100				