Department of Mathematics, University of Houston
Dr. Ronald H.W. Hoppe
Optimization Theory, Fall 2006

Optimization Theory
(7th Homework Assignment)

Exercise 14 *(Tangent and normal cone to the feasible set)*

Let the feasible set $\mathcal{F}$ be given by

$$\mathcal{F} \ = \ \{x \in \mathbb{R}^n \mid h(x) = 0\} \ ,$$

where $h : \mathbb{R}^n \to \mathbb{R}$ is smooth.
Prove that for $x \in \mathcal{F}$ with $\nabla h(x) \neq 0$ the tangent and the normal cone to $\mathcal{F}$ at x are given by

$$T_{\mathcal{F}}(x) \ = \ \mathrm{Ker}(\nabla h(x)^T) \quad , \quad N_{\mathcal{F}}(x) \ = \ \mathrm{Im}(\nabla h(x)^T) \ ,$$

where Ker and Im refer to the kernel and range, respectively.

4 points

Exercise 15 *(Tangent and normal cone to the feasible set)*

Consider the 'ice-cream cone'

$$\mathcal{F} \ := \ \{x \in \mathbb{R}^3 \mid x_3 \geq 2\sqrt{x_1^2 + x_2^2}\} \ .$$

At $x_0 = (0,0,0)^T$ the tangent cone $T_{\mathcal{F}}(x_0)$ coincides with the feasible set $\mathcal{F}$. Show that the normal cone is given by

$$N_{\mathcal{F}}(x_0) \ = \ \{v \in \mathbb{R}^3 \mid v_3 \leq -\frac{1}{2}\sqrt{v_1^2 + v_2^2}\} \ .$$

4 points

Exercise 16 *(Limiting directions and the tangent cone)*

Given $x^* \in \mathcal{F} := \{x \in \mathbb{R}^n \mid h_i(x) = 0, 1 \leq i \leq m, g_i(x) \leq 0, 1 \leq i \leq p\}$, the set

$$F_1 \ := \ \{ \ \alpha d \mid \alpha \geq 0 \ , \ \begin{array}{lcl} \nabla h_i(x^*)^T d \ = \ 0 & , & 1 \leq i \leq m \ , \\ \nabla g_i(x^*)^T d \ \leq \ 0 & , & i \in \mathcal{I}_{ac}(x^*) \end{array} \ \}$$

is a cone. Prove: If LICQ is satisfied, F_1 is the tangent cone to the feasible set $\mathcal{F}$ at x^*.

4 points

Delivery of the homework at latest on Oct. 18, 2006, 04:00 pm. The homework may be submitted either electronically (rohop@math.uh.edu) or as a hardcopy in class