Chapter 7: Pricing of Exotic Options

7.1 Exotic Options

American and European call and put options are called standard options. All other types of options are referred to as exotic options. The most important exotic options are:

Asian Options: The payoff of an Asian option depends on the average price of the basic asset.

Barrier Options: The payoff of a barrier option does not only depend on the value S_T of the asset at maturity date T, but on the path S_t of the asset an a certain threshold value B, called the barrier. The barrier B can be reached by the path S_t either from above (down) or from below (up). Moreover, the barrier option may terminate as soon as the barrier is reached (out) or come into effect (in). Therefore, we have the possibilities of down-and-in calls (puts), down-and-out calls (puts), up-and-in calls (puts), and up-and-out calls (puts).

For instance, the payoff in case of a down-and-out call is

$$V(S,T) = \begin{cases} (S_T - K)^+ & , \text{ if } S_t > B \text{ for all } 0 \le t \le T , \\ 0 & , \text{ if } S_t \le B \text{ for some } 0 \le t \le T \end{cases} .$$

Asian Options: The payoff of an Asian option depends on the average price of the basic asset.

Binary Options: The payoff of a binary option is discontinuous. For instance, given some $A > 0$, for a binary put we have

$$V(S,T) = \begin{cases} A & , \quad \text{if} \quad S_T < K \\ 0 & , \quad \text{if} \quad S_T \geq K \end{cases} .$$

Chooser options: At some prespecified time instant $0 \leq t_c < T$, the holder of the option decides whether the option is a call or a put. Therefore, we have

$$V(S,t_c) = \max \left(V_C(S,t_c), V_P(S,t_c) \right) .$$

Compound options: Compound options are options on options. We distinguish the cases: call on a call, call on a put, put on a call, put on a put.

Lookback options: The payoff of a lookback option depends on the minimum or maximum value of the asset S during its life time. For instance:

$$V(S,T) = \max \left(S_t \mid 0 \leq t \leq T \right) - S_T .$$

Path-Dependent Options: The payoff depends on the path of S_t. The most prominent examples of path-dependent options are Asian options, barrier options, and lookback options.

7.2 Asian Options with Continuous Averages

An Asian option can be based on an American or European call or put. The payoff depends either on the continuously sampled arithmetic average $\hat{S}_a$ or geometric average $\hat{S}_g$ of S_t

$$\hat{S}_a := \frac{1}{T}\int_0^T S_t \, dt \quad , \quad \hat{S}_g := \exp\left(\frac{1}{T}\int_0^T \log(S_t) \, dt\right) .$$

We distinguish between average price calls (puts) and average strike calls (puts) according to

$$V(S,T) = \begin{cases} (\hat{S}_{a/g} - K)^+ & \text{average price call} \\ (K - \hat{S}_{a/g})^+ & \text{average price put} \\ (S_T - \hat{S}_{a/g})^+ & \text{average strike call} \\ (\hat{S}_{a/g} - S_T)^+ & \text{average strike put} \end{cases} .$$

Note that price options are also called rate options or fixed strike options and strike options are sometimes referred to as floating strike options.

7.2.1 The Black-Scholes equation for Asian options

For an Asian option with a continuous average, the average can be written by means of

$$A_t = \int_0^t f(S_\tau, \tau)\, d\tau ,$$

where, for instance, $f(S_\tau, \tau) = S_\tau$ in case of the arithmetic average. Obviously, we have

$$dA_t = f(S_t, t)\, dt .$$

Considering a portfolio $Y_t := c_1(t)B_t + c_2(t)S_t - V(S_t, A_t, t)$ which we assume to be risk-free and self-financing, the application of Itô's lemma implies that the option price $V = V(S, A, t)$ satisfies the spatially two-dimensional parabolic partial differential equation

$$\frac{\partial V}{\partial t} + \frac{1}{2} \sigma^2 S^2 \frac{\partial^2}{\partial S^2} + r S \frac{\partial V}{\partial S} + f(S,t) \frac{\partial V}{\partial A} - rV = 0 \quad , \quad (S, A, t) \in Q := [0, \infty)^2 \times (0, T) ,$$

with final condition $V(S, A, T) = \Lambda(S, A)$ and boundary conditions $V(S, A, t) = V_b(S, A, t), (S, A) \in \partial\Omega, t \in (0, T)$, where $\Omega := [0, \infty)^2$ and $V_b(S, A, t)$ depends on the type of the Asian option.