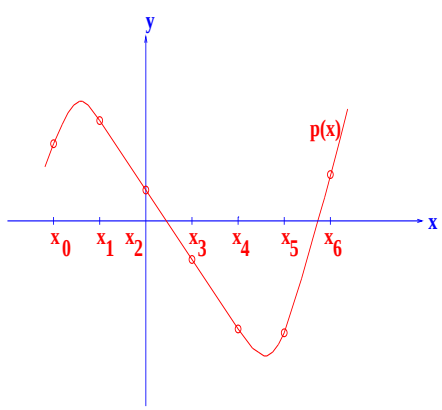
4 Interpolation

4.1 Polynomial interpolation

Problem: Let $P_n(I)$, $n \in \mathbb{N}$, $I := [a,b] \subset \mathbb{R}$, be the linear space of polynomials of degree $\leq n$ on I,

$$P_n(I) \;:=\; \{\, p_n : I \to \mathbb{R} \mid p_n(x) \;=\; \sum_{i=0}^{n} a_i\, x^i \;,\; a_i \in \mathbb{R} \;,\; 0 \leq i \leq n \,\} \;.$$

We have $\dim P_n(I) = n+1$, since any $p_n \in P_n(I)$ is uniquely determined by the coefficients a_i , $0 \leq i \leq n$.



Interpolation problem:

Given $n+1$ points (x_i, f_i) , $0 \leq i \leq n$, find $p_n \in P_n(I)$ such that

$$(+) \quad p_n(x_i) \;=\; f_i \;,\; 0 \leq i \leq n \;.$$

Definition 4.1 Interpolating polynomial

The points x_i (resp. f_i , $0 \leq i \leq n$,) are called nodal points (resp. nodal values) of the interpolation problem.
The $n+1$ equations $(+)$ are called the interpolating conditions.
A polynomial $p_n \in P_n(I)$ satisfying $(+)$ is called an interpolating polynomial.

4.1.1 Existence and Uniqueness

In explicit form, the interpolating conditions are given as follows:

$$\sum_{j=0}^{n} a_j \, x_i^j \;=\; f_i \quad , \quad 0 \le i \le n \, .$$

They represent a linear system in the coefficients a_j , $0 \le j \le n$,:

$$(\star) \qquad \underbrace{\begin{pmatrix} 1 & x_0 & x_0^2 & \cdot & \cdot & \cdot & x_0^n \\ 1 & x_1 & x_1^2 & \cdot & \cdot & \cdot & x_1^n \\ \cdot & \cdot & \cdot & & & & \cdot \\ \cdot & \cdot & \cdot & & & & \cdot \\ \cdot & \cdot & \cdot & & & & \cdot \\ 1 & x_n & x_n^2 & \cdot & \cdot & \cdot & x_n^n \end{pmatrix}}_{=:\, V} \begin{pmatrix} a_0 \\ a_1 \\ \cdot \\ \cdot \\ \cdot \\ \cdot \, a_n \end{pmatrix} = \begin{pmatrix} f_0 \\ f_1 \\ \cdot \\ \cdot \\ \cdot \\ \cdot \, f_n \end{pmatrix} .$$

Definition 4.2 Vandermonde's determinant
The coeffizient matrix V in $(\star)$ is called Vandermonde's matrix. Its determinant is referred to as Vandermonde's determinant.

Lemma 4.3 Vandermonde's determinant

Vandermonde's determinant is given by:

$$\det V = \prod_{0 \le j < k \le n} (x_k - x_j) .$$

Proof: We consider

$$V_n(x) := \begin{pmatrix} 1 & x_0 & x_0^2 & \cdots & x_0^n \\ 1 & x_1 & x_1^2 & \cdots & x_1^n \\ \cdot & \cdot & \cdot & & \cdot \\ \cdot & \cdot & \cdot & & \cdot \\ 1 & x_{n-1} & x_{n-1}^2 & \cdots & x_{n-1}^n \\ 1 & x & x^2 & \cdots & x^n \end{pmatrix} .$$

Expansion of the determinant w.r.t. the $(n+1, n+1)$-st element x^n results in:

$$\det V_n(x) = \underbrace{\det V_{n-1}(x_{n-1})}_{\text{leading coeff.}} x^n + q_{n-1} \quad , \quad q_{n-1} \in P_{n-1}(\mathbb{R}) .$$

We have:

$$\det V_n(x) = \underbrace{\det V_{n-1}(x_{n-1})}_{\text{leading coeff.}} x^n + q_{n-1} \quad , \quad q_{n-1} \in P_{n-1}(\mathbb{R}) .$$

On the other hand:

$$\det V_n(x) \in P_n(\mathbb{R}) \quad , \quad \det V_n(x_j) = 0 \, , \, 0 \le j \le n-1, \quad \Longrightarrow$$

$$\det V_n(x) = \det V_{n-1}(x_{n-1}) \prod_{j=0}^{n-1} (x - x_j) \quad \Longrightarrow$$

$$\det V = \det V_n(x_n) = \det V_{n-1}(x_{n-1}) \prod_{j=0}^{n-1} (x_n - x_j) =$$

$$= \det V_{n-2}(x_{n-2}) \prod_{j=0}^{n-2} (x_{n-1} - x_j) \prod_{j=0}^{n-1} (x_n - x_j) =$$

$$= \ldots = \underbrace{\det V(x_1)}_{=x_1 - x_0)} \prod_{1 \le j < k \le n} (x_k - x_j) .$$

Theorem 4.4 Existence and uniqueness of the interpolating polynomial

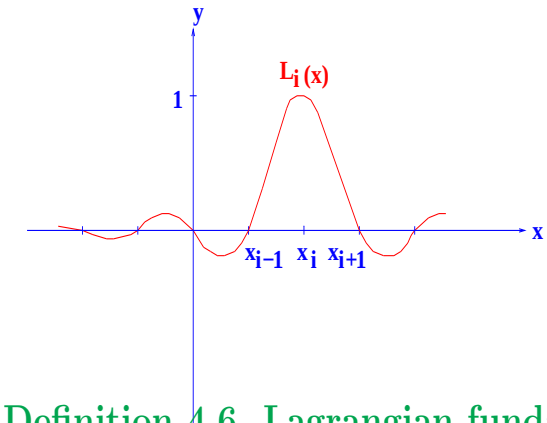
In case of mutually disjoint nodal points, i.e., $x_i \neq x_j$, $0 \leq i \neq j \leq n$, there exists a uniquely determined interpolating polynomial $p_n \in P_n(I)$.

Proof: There holds $\det \prod_{0 \leq j < k \leq n} (x_k - x_j) \neq 0$.

Remark 4.5 Computation of the interpolating polynomial

The computation of the polynomial by the solution of the linear system with the Vandermonde matrix as the coefficient matrix is not practicable, since the amount of computational work is too large. Moreover, the linear system is in general ill conditioned.

4.1.2 Lagrangian representation of the interpolating polynomial



In case of mutually disjoint nodal points x_i , $0 \le i \le n$, a basis of $P_n(\mathbb{R})$ is given by the polynomials

$$L_i \in P_n(\mathbb{R}) \quad , \quad L_i(x_j) \;=\; \delta_{ij} \;,\; 0 \le i,j \le n \;.$$

Obviously, there holds:

$$L_i(x_j) \;=\; 0 \;,\; i \neq j \quad \Longrightarrow \quad L_i(x) \;=\; \alpha_n \prod_{j \neq i}(x \;-\; x_j)$$

$$L_i(x_i) \;=\; \alpha_n \prod_{j \neq i}(x_i - x_j) \;=\; 1 \;\Longrightarrow\; \alpha_n \;=\; 1/\Big(\prod_{j \neq i}(x_i - x_j)\Big)$$

Definition 4.6 Lagrangian fundamental polynomials
The polynomials

$$L_i(x) \;=\; \prod_{j \neq i} \frac{x \;-\; x_j}{x_i \;-\; x_j}$$

are called Lagrangian fundamental polynomials.

Theorem 4.7 Lagrangian representation of the interpolating polynomial
In case of mutually disjoint nodal points x_i , $0 \le i \le n$, the interpolating polynomial admits the representation

$$p_n(x) = \sum_{i=0}^{n} f_i \, L_i(x) \, .$$

Remark 4.8 Practical computation of the interpolating polynomial
The computation of the interpolating polynomial by the Lagrangian representation formula is not practicable, since the addition of a pair (x_{n+1}, f_{n+1}) requires that all fundamental polynomials have to be recomputed.

4.1.3 Newton's representation of the interpolating polynomial

Idea: Recursive construction

Let $p_0 \in P_0(\mathbb{R})$ be the interpolating polynomial of degree 0 with respect to (x_0, f_0), i.e., $p_0(x) = f_0$.

Ansatz for $p_1 \in P_1(\mathbb{R})$ with respect to (x_0, f_0) , (x_1, f_1):

$$p_1(x) = p_0(x) + q_1(x) \quad , \quad q_1 \in P_1(\mathbb{R}) .$$

Determination of q_1:

$$p_1(x_0) = f_0 + q_1(x_0) = f_0 \implies q_0(x_0) = 0 ,$$
$$\implies q_1(x) = b_1 (x - x_0) \implies p_1(x) = f_0 + b_1 (x - x_0) .$$

We obtain b_1 from the condition $p_1(x_1) = f_1$.

Ansatz for $p_2 \in P_2(\mathbb{R})$ with respect to (x_0, f_0) , (x_1, f_1) , (x_2, f_2):

$$p_2(x) = p_1(x) + q_2(x) \quad , \quad q_2 \in P_2(\mathbb{R}) .$$

Determination of q_2:

$$p_2(x_0) = f_0 + q_2(x_0) = f_0 \implies q_2(x_0) = 0 \quad , \quad p_2(x_1) = f_1 + q_2(x_1) = f_1 \implies q_2(x_1) = 0$$
$$\implies q_2(x) = b_2 (x - x_0)(x - x_1) .$$

General case: Let $p_{n-1} \in P_{n-1}(\mathbb{R})$ be the interpolating polynomial with respect to (x_i, f_i) , $0 \le i \le n-1$.

Ansatz for $p_n \in P_n(\mathbb{R})$ with respect to (x_i, f_i) , $0 \le i \le n-1$:

$$p_n(x) = p_{n-1}(x) + q_n(x) \quad , \quad q_n \in P_n(\mathbb{R}) .$$

Determination of q_n:

$$p_n(x_i) = f_i + q_n(x_i) = f_i \implies q_n(x_i) = 0 , \; 0 \le i \le n-1$$
$$\implies q_n(x) = b_n (x - x_0)(x - x_1)\ldots(x - x_{n-1}) .$$

In principle, b_n can be obtained from the condition:

$$p_n(x_n) = p_{n-1}(x_n) + q_n(x_n) = f_n .$$

However, this approach is not practicable, since we must determine p_{n-1} and evaluate it in x_n.
On the other hand, we have:

$$p_n(x) = b_n x^n + \sum_{i=0}^{n-1} c_i x^i .$$

b_n is the leading coefficient, i.e., the coefficient of x^n in p_n.
Most wanted: Formula for the computation of the leading coefficients.

Definition 4.9 Newton's representation of the interpolating polynomial

The representation of the interpolating polynomial given by

$$p_n(x) = b_0 + b_1 (x - x_0) + \ldots + b_n (x - x_0) \ldots (x - x_{n-1})$$

is called Newton's representation.

Recursive computation of the leading coefficients:

Let $p_{ij} \in P_{j-i}(\mathbb{R})$, $0 \le i < j \le n$, be the interpolating polynomials with respect to (x_i, f_i) , (x_{i+1}, f_{i+1}) , ..., (x_j, f_j). Ansatz for p_{ij}:

$$p_{ij}(x) = r_1(x)\, p_{i,j-1}(x) + s_1(x)\, p_{i+1,j}(x) \quad , \quad r_1, s_1 \in P_1(\mathbb{R}) .$$

Determination of r_1, s_1 :

$$p_{ij}(x_i) = r_1(x_i)\, p_{i,j-1}(x_i) + s_1(x_i)\, p_{i+1,j}(x_i) = f_i$$

$$\text{Set} \quad r_1(x_i) = 1 \,,\; s_1(x_i) = 0 \,,$$

$$p_{ij}(x_j) = r_1(x_j)\, p_{i,j-1}(x_j) + s_1(x_j)\, p_{i+1,j}(x_j) = f_j$$

$$\text{Set} \quad r_1(x_j) = 0 \,,\; s_1(x_j) = 1$$

$$\Longrightarrow \quad r_1(x) = \frac{x_j - x}{x_j - x_i} \,,\; s_1(x) = \frac{x - x_i}{x_j - x_i} .$$

Theorem 4.10 Recursion formula for the interpolating polynomials

The polynomials $p_{ij} \in P_{j-i}(\mathbb{R})$, $0 \le i < j \le n$, interpolating in (x_k, f_k) , $i \le k \le j$, satisfy the recursion

$$p_{ij}(x) = \frac{x_j - x}{x_j - x_i}\, p_{i,j-1}(x) + \frac{x - x_i}{x_j - x_i}\, p_{i+1,j}(x) .$$

Corollary 4.11 Recursion formula for the leading coefficients

The leading coefficients $f[x_i, x_{i+1}, \ldots, x_j]$ of the interpolating polynomials $p_{ij} \in P_{j-i}(\mathbb{R}), 0 \le i < j \le n$, satisfy the recursion

$$(*) \quad f[x_i, x_{i+1}, \ldots, x_j] = \frac{f[x_{i+1}, \ldots, x_j] - f[x_i, x_{i+1}, \ldots, x_{j-1}]}{x_j - x_i} \quad \text{where } f[x_i] := f_i \ , \ 0 \le i \le n .$$

Example 4.12 $(i = 0 \ , \ j = 1)$

$$f[x_0, x_1] = \frac{f[x_1] - f[x_0]}{x_1 - x_0} = \frac{f_1 - f_0}{x_1 - x_0} .$$

Definition 4.13 Divided differences

The quantities defined recursively by means of $(*)$ are called divided differences.
In particular, $f[x_i, \ldots, x_j]$ is said to be the $(j-i)$-th divided difference.

Remark 4.14 Recursive scheme for the divided differences

$$
\begin{array}{c|cccccc}
x_0 & f_0 = f[x_0] & & & & & \\
 & & f[x_0,x_1] & & & & \\
x_1 & f_1 = f[x_1] & & f[x_0,x_1,x_2] & & & \\
 & & f[x_1,x_2] & & \cdot & & \\
x_2 & f_2 = f[x_2] & & & \cdot & & \\
\cdot & \cdot & & & & \cdot & \\
\cdot & \cdot & & & & & f[x_0,\ldots,x_n] \\
\cdot & \cdot & & & & \cdot & \\
x_{n-2} & f_{n-2} = f[x_{n-1}] & & & \cdot & & \\
 & & f[x_{n-2},x_{n-1}] & & \cdot & & \\
x_{n-1} & f_{n-1} = f[x_{n-1}] & & f[x_{n-2},x_{n-1},x_n] & & & \\
 & & f[x_{n-1},x_n] & & & & \\
x_n & f_n = f[x_n] & & & & &
\end{array}
$$

Lemma 4.15 Newton's representation with divided differences
Since the coefficients b_i , $0 \le i \le n$, of Newton's interpolating polynomial correspond to the divided differences $f[x_0, ..., x_i]$, we obtain the following representation

$$(\star) \quad p_n(x) = f[x_0] + f[x_0, x_1](x - x_0) + ... + f[x_0, ..., x_n](x - x_0) ... (x - x_{n-1}) .$$

4.1.4 Evaluation of interpolating polynomials

(i) Horner's scheme

We reformulate $(\star)$ according to

$$p_n(x) = (f[x_0, ..., x_n](x - x_{n-1}) + f[x_0, ..., x_{n-1}](x - x_{n-2}) + ... + f[x_0, x_1](x - x_0)) + f[x_0] .$$

This leads to Horner's scheme:

Compute	$f[x_0, ..., x_n](x - x_{n-1}) + f[x_0, ..., x_{n-1}]$
Multiply by	$(x - x_{n-2})$ and add $f[x_0, ..., x_{n-2}]$
Multiply by	$(x - x_{n-3})$ and add $f[x_0, ..., x_{n-3}]$
etc.	

Disadvantage: Explicit knowledge of Newton's interpolating polynomial is required.
Too much computational work in case of evaluation at a single point.

(ii) Algorithm of Aitken-Neville

Idea: Application of the recursion formula for the interpolating polynomials:

$$
\begin{array}{c|ccccccc}
x_0 & f_0 = p_{00}(x) & & & & & & \\
 & & p_{01}(x) & & & & & \\
x_1 & f_1 = p_{11}(x) & & p_{02}(x) & & & & \\
 & & p_{01}(x) & & \cdot & & & \\
x_2 & f_2 = p_{22}(x) & & & \cdot & & & \\
\cdot & \cdot & & & & \cdot & & \\
\cdot & \cdot & & & & & p_{0n}(x) & \\
\cdot & \cdot & & & & \cdot & & \\
x_{n-2} & f_{n-2} = p_{n-2,n-2}(x) & & & \cdot & & & \\
 & & p_{n-2,n-1}(x) & & \cdot & & & \\
x_{n-1} & f_{n-1} = p_{n-1,n-1}(x) & & p_{n-2,n}(x) & & & & \\
 & & p_{n-1,n}(x) & & & & & \\
x_n & f_n = p_{nn}(x) & & & & & &
\end{array}
$$

4.1.5 Remainder term and error estimate

Problem: Consider the approximation of a continuous function $f : I := [a, b] \to \mathbb{R}$ by the polynomial $p_n \in P_n(I)$ interpolating in $x_i \in [a, b]$, $0 \le i \le n$,.
What can be said about the accuracy of this approximation?

Definition 4.16 Remainder term, interpolation error

Assume $f \in C([a, b])$ and let $p_n \in P_n([a, b])$ be the polynomial interpolating f in $(x_i, f(x_i))$, $x_i \in [a, b]$, $0 \le i \le n$. The function $R_n := f - p_n$ is called the remainder of the interpolation. The value of the remainder in $x \in [a, b]$, $x \ne x_i$, $0 \le i \le n$, is referred to as the interpolation error.

Theorem 4.17 Representation of the remaining term

Let $f \in C^{n+1}([a,b])$ and assume that $p_n \in P_n([a,b])$ is the polynomial interpolating f in $(x_i, f(x_i))$, $x_i \in [a,b], 0 \le i \le n$. For any $x \in [a,b]$ there exists $\xi \in I_x := [\min(x_0, ..., x_n, x), \max(x_0, ..., x_n, x)\ ]$ such that

$$(*) \quad R_n(x) = f(x) - p_n(x) = \frac{f^{(n+1)}(\xi)}{(n+1)!}\,\omega_{n+1}(x) \quad , \quad \omega_{n+1}(x) := (x - x_0)\,(x - x_1)\,...\,(x - x_n)\,.$$

Proof: W.r.o.g. let $x \neq x_i$, $0 \le i \le n$. We consider the auxiliary function

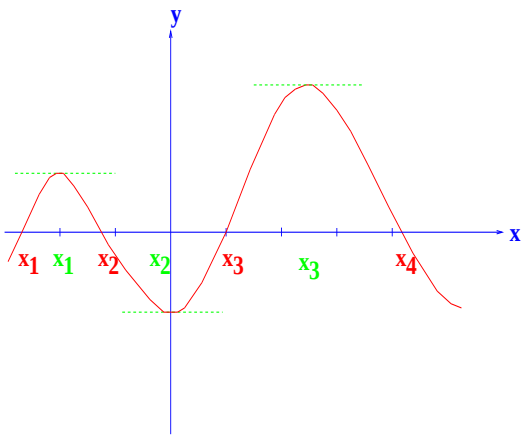
$$F(t) := f(t) - p_n(t) - \frac{f(x) - p_n(x)}{\omega_{n+1}(x)}\,\omega_{n+1}(t)\ ,\ t \in [a,b]\ .$$

Obviously, we have $F \in C^{n+1}([a,b])$, and there holds

$$F(x_i) = 0\ ,\ 0 \le i \le n \quad , \quad F(x) = 0\ .$$

$\Longrightarrow$ F has at least $n+2$ zeroes in $[a,b]$.

We already know: F has at least $n+2$ zeroes in $[a,b]$.



A repeated application of Rolle's theorem yields:

F' has at least $n+1$ zeroes in $[a,b]$,

..... ,

$F^{(n+1)}$ has at least 1 zero in $\xi \in [a,b]$.

$$F^{(n+1)}(\xi) = f^{(n+1)}(\xi) - \underbrace{p_n^{(n+1)}(\xi)}_{=\,0} - \frac{f(x)-p_n(x)}{\omega_{n+1}(x)}\,(n+1)! = 0$$

$$\Longrightarrow \quad f(x) - p_n(x) = \frac{f^{(n+1)}(\xi)}{(n+1)!}\,\omega_{n+1}(x) \quad .$$

Corollary 4.18 Representation of the remaining term using divided differences
Under the assumptions of Theorem 4.17 there holds:

$$f[x_0,\ldots,x_n,x] = \frac{f^{(n+1)}(\xi)}{(n+1)!} \quad , \quad \xi \in [a,b] \, .$$

Proof: There holds:

$$p_n(x) = p_{0,n}(x) = \sum_{i=0}^{n} f[x_0, \ldots, x_i] \prod_{j=0}^{i-1} (x - x_j) .$$

Setting $x_{n+1} := x$ and $f_{n+1} := f(x)$, we obtain:

$$f(x) = p_{0,n+1}(x) = p_{0,n}(x) + f[x_0, \ldots, x_n, x]\, \omega_{n+1}(x) .$$

Corollary 4.19 Generalized mean value theorem

If $f \in C^n([a,b])$, then there exists $\xi \in [\, \min(x_0, \ldots, x_n), \max(x_0, \ldots, x_n)\,]$ such that

$$f[x_0, \ldots, x_n] = \frac{f^{(n)}(\xi)}{n!} .$$

Corollary 4.21 Estimate of the remainder

If $f \in C^{n+1}([a,b])$ and if $p_n \in P_n([a,b])$ denotes the polynomial interpolating f in $x_i \in [a,b]$, $0 \le i \le n$,, then the following estimate holds true

$$|\, f(x) - p_n(x) \,| \le \max_{\xi \in [a,b]} \frac{|f^{(n+1)}(\xi)|}{(n+1)!} \max_{x \in [a,b]} |\omega_{n+1}(x)| .$$

Problem: Chebyshev nodes

The estimate

$$|\, f(x) \;-\; p_n(x) \,| \;\le\; \max_{\xi\in[a,b]} \frac{|f^{(n+1)}(\xi)|}{(n+1)!} \max_{x\in[a,b]} |\omega_{n+1}(x)| \,.$$

motivates to determine the nodes $x_i \in [a,b]$, $0 \le i \le n$, such that

$$(\diamond) \quad \max_{x\in[a,b]} |\, \omega_{n+1}(x) \,| \;=\; \max_{x\in[a,b]} |\, (x \;-\; \bar{x}_0)\ldots(x \;-\; \bar{x}_n) \,| \;=\; \min_{x_i\in[a,b]} \max_{x\in[a,b]} |\, (x \;-\; x_0)\ldots(x \;-\; x_n) \,|$$

$(\diamond)$ represents a minimax-problem whose solution is already known from the cg method

Theorem 4.21 Chebyshev nodes

The solution of the minimax-problem $(\diamond)$ is given by the zeroes

$$t_i \;=\; \cos\Big(\frac{2i+1}{2n+2}\,\pi\Big) \quad , \quad 0 \le i \le n$$

of the Chebyshev polynomial $T_{n+1}(x) = \cos((n+1)\arccos(x))$ transformed to the interval $[a,b]$.

We thus obtain the Chebyshev nodes

$$\bar{x}_i \;=\; \frac{b \;-\; a}{2}\, t_i \;+\; \frac{b \;+\; a}{2} \quad , \quad 0 \le i \le n \,.$$

4.1.6 Convergence of interpolating polynomials

Remark 4.22 About Weierstrass' theorem

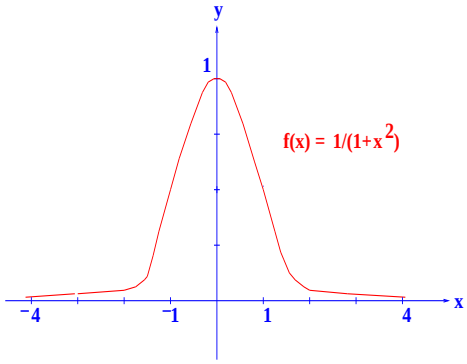
Weierstrass' theorem states: If $f : [a, b] \to \mathbb{R}$ is continuous, for any $\varepsilon > 0$ there exists a polynomial p_ε such that

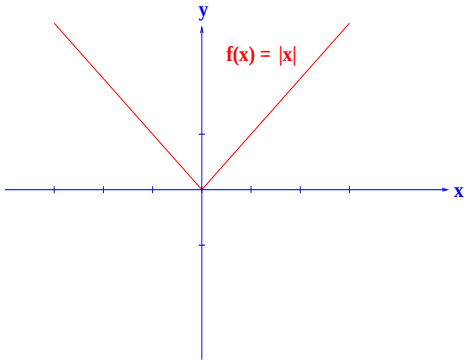
$$\sup_{x \in [a,b]} |f(x) - p_\varepsilon(x)| \leq \varepsilon .$$

Weierstrass' theorem does not imply convergence of the sequence of interpolating polynomials.

Problem: Assume that $f : [a, b] \to \mathbb{R}$ is a continuous function, $x_i := a + i \frac{b-a}{n}$, $0 \leq i \leq n$, and let $p_n \in P_n([a, b])$ be the polynomials interpolating in $(x_i, f(x_i))$, $0 \leq i \leq n$.

What can be said about the convergence of the sequence $(p_n)_{n \in \mathbb{N}}$ of the interpolating polynomials?





(i) Counterexample by Runge (1901)

$$f(x) = \frac{1}{1+x^2} \quad , \quad x \in [-5,+5] \ .$$

The sequence of interpolating polynomials $(p_n)_{n\in\mathbb{N}}$ with respect to equidistant nodes x_i , $0 \le i \le n$, converges for $|x| \le 3.63$ and diverges for $3.63 < |x| < 5$.

(ii) Counterexample by Bernstein (1912)

$$f(x) = |x| \quad , \quad x \in [-1,+1] \ .$$

The sequence of interpolating polynomials $(p_n)_{n\in\mathbb{N}}$ diverges for all $0 < |x| < 1$.

Definition 4.23 Nodal scheme
Let $x_{n,i}$, $0 \le i \le n$, $n \in \mathbb{N}_0$, be the mutually different nodes of $p_n \in P_n([a,b])$ such that $p_n(x_{n,i}) = f(x_{n,i})$, $0 \le i \le n$. Then, the scheme

$$S \; : \; \begin{array}{lllll} x_{0,0} & & & & \\ x_{1,0} & x_{1,1} & & & \\ \cdot & \cdot & \cdot & & \\ x_{n,0} & x_{n,1} & \cdots & & x_{n,n} \\ \cdot & \cdot & \cdots & \cdot & \cdot \end{array}$$

is called the nodal scheme of the sequence $(p_n)_{n\in\mathbb{N}_0}$ of interpolating polynomials.

Theorem 4.24 Theorem of Marcinkiewicz
For any function $f \in C([a,b])$ there exists a nodal scheme S with $x_{n,i} \in [a,b]$, $0 \le i \le n, n \in \mathbb{N}_0$, such that the sequence $(p_n)_{n\in\mathbb{N}_0}$ of interpolating polynomials uniformly converges to f.

Remark 4.25 About the theorem of Marcienkiewicz
The proof of the theorem of Marcinkiewicz uses nonconstructive results from approximation theory and hence, it does not provide a practical method for the determination of the nodal scheme.

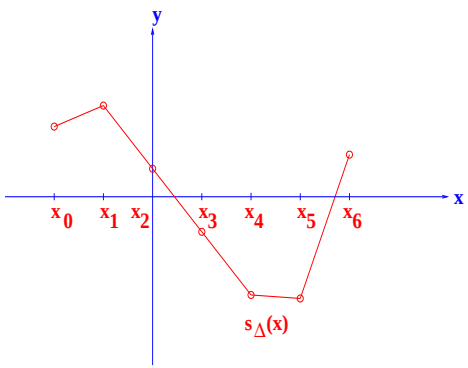
Theorem 4.26 Faber's theorem

For any nodal scheme S with $x_{n,i} \in [a,b]$, $0 \le i \le n$, $n \in \mathbb{N}_0$, there exists a continuous function $f : [a,b] \to \mathbb{R}$ such that the sequence $(p_n)_{n \in \mathbb{N}_0}$ of interpolating polynomials does not uniformly converge to f.

4.2 Spline interpolation

4.2.1 Foundations

Example 4.27 Interpolation by piecewise linear functions



Assume that $\Delta := \{x_0, ..., x_n\}$ is a grid of $n+1$ mutually disjoint nodes

$$a := x_0 \ < \ x_1 \ < \ ... \ < \ x_n =: b \ .$$

Given the values f_i , $0 \le i \le n$, consider the interpolating function s_Δ with

$$s_\Delta|_{[x_i, x_{i+1}]} \in P_1([x_i, x_{i+1}]) \ , \ 0 \le i \le n-1 \ , \ s_\Delta(x_i) \ = \ f_i \ , \ 0 \le i \le n \ .$$

Obviously, s_Δ is continuous, but not continuously differentiable. The critical points are the inner nodes x_i , $1 \le i \le n-1$.

Idea: Use piecewise polynomials of a higher degree with 'better' global smoothness properties.

Definition 4.28 Spline of degree $m-1$

Assume that $\Delta := \{x_0, ..., x_n\}$ is a grid of $n+1$ mutually disjoint nodes satisfying

$$a := x_0 < x_1 < ... < x_n =: b .$$

A function $s_\Delta : [a,b] \to \mathbb{R}$ is called a spline of degree $m-1$, $m \in \mathbb{N}$, (resp. of order m) with respect to Δ, if

$$(S_1) \quad s_\Delta \in C^{m-2}([a,b]) \quad (m \geq 2),$$
$$(S_2) \quad s_i := s_\Delta|_{[x_i,x_{i+1}]} \in P_{m-1}([x_i,x_{i+1}]) , \; 0 \leq i \leq n-1 .$$

The splines of degree $m-1$ form a linear space which will be denoted by $S_{m,\Delta}$.

In particular, we obtain

$m = 1$ step functions

$m = 2$ polygons (linear splines)

$m = 3$ quadratic splines

$m = 4$ cubic splines

Remark 4.29 Dimension

A spline of the order m is the $m-1$-fold integral of a spline of order 1 (step function):

$$\dim S_{1,\Delta} = n \quad \Longrightarrow \quad \dim S_{m,\Delta} = n + m - 1 .$$

Continuity and differentiability conditions in the inner nodes:

$$s_{i-1}^{(k)}(x_i) = s_i^{(k)}(x_i) \quad , \quad 1 \le i \le n-1 \, , \, 0 \le k \le m-2 .$$

By $m-1$-fold integration we obtain:

$$s_i(x) = s_{i-1}(x) + \gamma_i \, (x - x_i)^{m-1} ,$$

and hence,

$$s_\Delta(x) = \begin{cases} s_{i-1}(x) & , \quad x < x_i \\ s_{i-1}(x) + \gamma_i \, (x - x_i)^{m-1} & , \quad x \ge x_i \end{cases} .$$

Definition 4.30 Truncated power function

The function given by

$$(x - x_i)_+^{m-1} := \begin{cases} 0 & , \quad x < x_i \\ (x - x_i)^{m-1} & , \quad x \ge x_i \end{cases}$$

is called truncated power function.

Remark 4.31 Differentiation of truncated power functions

$$\frac{d}{dx}(x - x_i)_+^{m-1} = (m - 1)(x - x_i)_+^{m-2} \quad (m \geq 2) ,$$

$$\frac{d}{dx}(x - x_i)_+ = \begin{cases} 0 & , \quad x < x_i \\ 1 & , \quad x \geq x_i \end{cases} \quad (m = 2) .$$

Theorem 4.32 Basis of $S_{m,\Delta}$

The linear space $S_{m,\Delta}$, $m \in \mathbb{N}$, of splines of order m satisfies:

$$(+) \qquad S_{m,\Delta} = \text{span}\,\{1, x, ..., x^{m-1}, (x - x_1)_+^{m-1}, ..., (x - x_{n-1})_+^{m-1}\} ,$$

$$(++) \qquad \dim S_{m,\Delta} = n + m - 1 .$$

Proof: We have to verify the linear independence of the functions in $(+)$. Assume that:

$$s_\Delta(x) = \sum_{k=0}^{m-1} a_k x^k + \sum_{j=1}^{n-1} c_j (x - x_j)_+^{m-1} = 0 \quad , \quad \text{for all } x \in [a, b] .$$

It has to be shown that

$$a_k = 0 , \; 0 \leq k \leq m - 1 \quad , \quad c_j = 0 , \; 1 \leq j \leq n - 1 .$$

Consider the linear functionals

$$G_i(f) \;=\; \frac{1}{(m-1)!}\,\left[\, f^{(m-1)}(x_{i+0}) \;-\; f^{(m-1)}(x_{i-0}) \,\right] \quad , \quad 1 \le i \le n-1 \,.$$

Inserting s_Δ for f, we obtain:

$$G_i(s_\Delta) \;=\; \underbrace{G_i\Big(\sum_{k=0}^{m-1} a_k\, x^k\Big)}_{=0} \;+\; \sum_{j=1}^{n-1} c_j\, \underbrace{G_i\big((x \;-\; x_j)_+^{m-1}\big)}_{=\delta_{ij}} \;=\; c_i \;=\; 0\,,\; 1 \le i \le n-1\,.$$

It follows that

$$s_\Delta(x) \;=\; \sum_{k=0}^{m-1} a_k\, x^k \;=\; 0\,,\; x \in [a,b] \quad \Longrightarrow \quad a_k \;=\; 0\,,\; 0 \le k \le m-1\,.$$

Remark 4.33 Aspects of the basis representation

The evaluation of a spline in the basis representation

$$s_\Delta(x) \;=\; \sum_{k=0}^{m-1} a_k\, x^k \;+\; \sum_{j=1}^{n-1} c_j\, (x \;-\; x_j)_+^{m-1}$$

can not be recommended, since

- it is ill conditioned with respect to perturbations in c_j ,
- the basis functions are non-local and the coefficients a_k, c_j do not have a geometrical meaning.

4.3.2 The cubic spline interpolant

Definition 4.34 Cubic spline interpolant

Assume that

$$\Delta := \{ a = x_0 < x_1 < \ldots < x_n = b \}$$

is a not necessarily equidistant partition of the interval $[a, b]$ and $f_i \in \mathbb{R}, 0 \le i \le n$.
A spline $s_\Delta \in S_{4,\Delta}$ is called a cubic spline interpolant, if

$$(*) \quad s_\Delta(x_i) = f_i \quad , \quad 0 \le i \le n \, .$$

Remark 4.35 The well-posedness of the interpolation problem

A simple reasoning reveals that a cubic spline interpolant is not uniquely determined alone by the interpolating conditions $(*)$:

Number of unknowns:

4 (unknowns per subinterval) $\times$ n (number of subintervals) $= 4n$

Number of determining equations:

3 (continuity and differentiability) $\times$ $(n-1)$ (Number of interior nodes)
$+ (n+1)$ interpolating conditions $= 4n - 2$.

Definition 4.36 Complete, natural, and periodic cubic spline interpolant
A cubic spline interpolant is called

$$\left.\begin{array}{r}\text{complete,}\\ \text{natural,}\\ \text{periodic,}\end{array}\right\} \quad \text{if} \quad \left\{\begin{array}{rcl} s'_\Delta(x_0) = f'_0 & , & s'_\Delta(x_n) = f'_n\,, \\ s''_\Delta(x_0) = 0 & , & s''_\Delta(x_n) = 0\,, \\ s'_\Delta(x_0) = s'_\Delta(x_n) & , & s''_\Delta(x_0) = s''_\Delta(x_n)\,. \end{array}\right.$$

Remark 4.37 Existence, uniqueness, and computation
The proof of the existence and uniqueness of the complete (natural, periodic) cubic spline interpolant is constructive and thus provides a method of computation. There are two techniques:

- Computation via the slopes,
- Computation via the moments.

(i) Computation via the slopes

In each subinterval $[x_i, x_{i+1}]$, $0 \le i \le n-1$, the polynomials $s_i = s_\Delta|_{[x_i,x_{i+1}]}$ admit a representation as solutions of the Hermite interpolation problem:
Compute $s_i \in P_3([x_i, x_{i+1}])$ as given by

$$(\diamond) \quad s_i(x) = a_0 + a_1 (x - x_i) + a_2 (x - x_i)^2 + a_3 (x - x_i)^3 ,$$

such that

$$(+) \qquad s_i(x_j) = f_j \quad , \quad i \le j \le i+1 ,$$
$$(++) \qquad s_i'(x_j) = m_j \quad , \quad i \le j \le i+1 .$$

Here, the slopes m_i in the nodes are unknown and have to be computed by means of the determining equations.

The coefficients a_k , $0 \le k \le 3$, in $(\diamond)$ can be uniquely determined by the conditions $(+)$ and $(++)$.
Setting $h_i := x_{i+1} - x_i$, we obtain:

$$(\star) \quad s_i(x) = f_i + m_i (x - x_i) + \left[3 \frac{f_{i+1} - f_i}{h_i^2} - \frac{m_{i+1} + 2m_i}{h_i} \right] (x - x_i)^2 +$$
$$+ \left[-2 \frac{f_{i+1} - f_i}{h_i^3} + \frac{m_{i+1} + m_i}{h_i^2} \right] (x - x_i)^3 .$$

The function given by

$$(\star) \quad s_i(x) = f_i + m_i\,(x - x_i) + \left[\, 3\,\frac{f_{i+1} - f_i}{h_i^2} - \frac{m_{i+1} + 2m_i}{h_i} \,\right](x - x_i)^2 +$$
$$+ \left[\, -2\,\frac{f_{i+1} - f_i}{h_i^3} + \frac{m_{i+1} + m_i}{h_i^2} \,\right](x - x_i)^3$$

satisfies $s_\Delta \in C^1([a,b])$. The condition $s_\Delta \in C^2([a,b])$ leads to

$$(\bullet) \quad s_{i-1}^{(2)}(x_i) = s_i^{(2)}(x_i) \quad , \quad 1 \le i \le n-1 \,.$$

Differentiation in $(\star)$ results in

$$s_i^{(2)}(x) = 2\left[\, 3\,\frac{f_{i+1} - f_i}{h_i^2} - \frac{m_{i+1} + 2m_i}{h_i} \,\right] + 6\left[\, -2\,\frac{f_{i+1} - f_i}{h_i^3} + \frac{m_{i+1} + m_i}{h_i^2} \,\right](x - x_i) \,.$$

Hence, $(\bullet)$ provides the equations

$$h_i\, m_{i-1} + 2\,(h_{i-1} + h_i)\, m_i + h_{i-1}\, m_{i+1} = 3\left[\, h_{i-1}\,\frac{f_{i+1} - f_i}{h_i} + h_i\,\frac{f_i - f_{i-1}}{h_{i-1}} \,\right] .$$

If we divide in

$$h_i\, m_{i-1} \;+\; 2\,(h_{i-1} \;+\; h_i)\, m_i \;+\; h_{i-1}\, m_{i+1} \;=\; 3\,\Big[\, h_{i-1}\,\frac{f_{i+1}-f_i}{h_i} \;+\; h_i\,\frac{f_i-f_{i-1}}{h_{i-1}}\,\Big]\,.$$

by $h_{i-1}+h_i>0$, we obtain:

$$\underbrace{\frac{h_i}{h_{i-1}+h_i}}_{=:\,\rho_i}\, m_{i-1} \;+\; 2\, m_i \;+\; \underbrace{\frac{h_{i-1}}{h_{i-1}+h_i}}_{=:\,\lambda_i}\, m_{i+1} \;=\; \underbrace{\frac{3}{h_{i-1}+h_i}\,\Big[\, h_{i-1}\,\frac{f_{i+1}-f_i}{h_i} \;+\; h_i\,\frac{f_i-f_{i-1}}{h_{i-1}}\,\Big]}_{=:\,d_i}\,.$$

These are $n-1$ equations in the $n+1$ unknowns m_i , $0 \le i \le n$.

(i) Complete cubic spline interpolant

In case of a complete cubic spline interpolant, we additionally obtain:

$$\left.\begin{array}{rcl} s'_{\Delta}(x_0) &=& f'_0 \\ s'_{\Delta}(x_n) &=& f'_n \end{array}\right\} \quad\Longrightarrow\quad \left\{\begin{array}{rcl} m_0 &=& f'_0 \\ m_n &=& f'_n \end{array}\right. .$$

For the remaining $n-1$ unknowns m_i , $1 \le i \le n-1$, we obtain the linear tridiagonal system

$$\begin{pmatrix} 2 & \lambda_1 & 0 & \cdot & \cdot & \cdot & 0 \\ \rho_2 & 2 & \lambda_2 & \cdot & \cdot & \cdot & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ 0 & \cdot & \cdot & \cdot & \rho_{n-2} & 2 & \lambda_{n-2} \\ 0 & \cdot & \cdot & \cdot & 0 & \rho_{n-1} & 2 \end{pmatrix} \begin{pmatrix} m_1 \\ m_2 \\ \cdot \\ \cdot \\ m_{n-2} \\ m_{n-1} \end{pmatrix} = \begin{pmatrix} d_1 - \rho_1 f_0' \\ d_2 \\ \cdot \\ \cdot \\ d_{n-2} \\ d_{n-1} - \lambda_{n-1} f_n' \end{pmatrix} .$$

In view of

$$\rho_i + \lambda_i = \frac{h_i}{h_{i-1} + h_i} + \frac{h_{i-1}}{h_{i-1} + h_i} = 1 \quad , \quad 1 \le i \le n-1$$

the coefficient matrix is strictly diagonally dominant and hence, the linear system is uniquely solvable.

(ii) Natural cubic spline interpolant

In this case, we have the additional boundary conditions

$$s''_{\Delta}(x_0) = 0 \quad , \quad s''_{\Delta}(x_n) = 0 \, .$$

Consequently, we obtain the additional two equations

$$2\, m_0 + m_1 = 3\, \frac{f_1 - f_0}{h_0} =: d_0 \, ,$$

$$m_{n-1} + 2\, m_n = 3\, \frac{f_n - f_{n-1}}{h_{n-1}} =: d_n \, .$$

The coefficient matrix of the resulting linear system of equations also is a strictly diagonally dominant tridiagonal matrix.

(ii) Periodic cubic spline interpolant

Considering the additional boundary conditions

$$s'_\Delta(x_0) = s'_\Delta(x_n) \quad , \quad s''_\Delta(x_n) = s''_\Delta(x_n) \, ,$$

the first equation readily implies $m_0 = m_n$. The second equation yields

$$\frac{2}{h_0} m_0 + \frac{1}{h_0} m_1 + \frac{1}{h_{n-1}} m_{n-1} + \frac{2}{h_{n-1}} m_n = 3 \left[\frac{f_1 - f_0}{h_0^2} + \frac{f_n - f_{n-1}}{h_{n-1}^2} \right] =: d'_0 \, .$$

In view of $m_n = m_0$, it follows that

$$\left(\frac{2}{h_0} + \frac{2}{h_{n-1}}\right) m_0 + \frac{1}{h_0} m_1 + \frac{1}{h_{n-1}} m_{n-1} = d'_0 \, .$$

The remaining $n-1$ equations read as follows

$$\frac{1}{h_{i-1}} m_{i-1} + \left(\frac{2}{h_{i-1}} + \frac{2}{h_i}\right) m_i + \frac{1}{h_i} m_{i+1} = \left(\frac{1}{h_{i-1}} + \frac{1}{h_i}\right) d_i =: d'_i \quad , \quad 1 \le i \le n-2 \, ,$$

$$\frac{1}{h_{n-1}} m_0 + \frac{1}{h_{n-2}} m_{n-2} + \left(\frac{2}{h_{n-2}} + \frac{2}{h_{n-1}}\right) m_{n-1} = \left(\frac{1}{h_{n-2}} + \frac{1}{h_{n-1}}\right) d_{n-1} =: d'_{n-1} \, .$$

The coefficient matrix of the resulting linear algebraic system is given by

$$\begin{pmatrix} \left(\frac{2}{h_0}+\frac{2}{h_{n-1}}\right) & \frac{1}{h_0} & 0 & 0 & \cdot & \cdot & \frac{1}{h_{n-1}} \\ \frac{1}{h_0} & \left(\frac{2}{h_0}+\frac{2}{h_1}\right) & \frac{1}{h_1} & 0 & \cdot & \cdot & 0 \\ 0 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & 0 \\ 0 & \cdot & \cdot & \cdot & \frac{1}{h_{n-3}} & \left(\frac{2}{h_{n-3}}+\frac{2}{h_{n-2}}\right) & \frac{1}{h_{n-2}} \\ \frac{1}{h_{n-1}} & 0 & \cdot & \cdot & 0 & \frac{1}{h_{n-2}} & \left(\frac{2}{h_{n-2}}+\frac{2}{h_{n-1}}\right) \end{pmatrix}.$$

Obviously, the coefficient matrix also is strictly diagonally dominant.

Theorem 4.38 Existence and uniqueness of the cubic spline interpolant
The complete (resp. natural, periodic) cubic spline interpolant $s_\Delta \in S_{4,\Delta}$ is uniquely determined.
The slopes $m_1, \ldots, m_{n-1}$ (resp. $m_0, \ldots, m_n$) are the solutions of a linear algebraic system with a strictly diagonally dominant coefficient matrix.

(ii) Computation via the moments

In each subinterval $[x_i, x_{i+1}]$, $0 \le i \le n-1$, the polynomials $s_i = s_\Delta|_{[x_i,x_{i+1}]}$ admit the representation as the solution of the local interpolation problem:
Compute $s_i \in P_3([x_i, x_{i+1}])$ as given by

$$(\diamond) \quad s_i(x) = a_0 + a_1 (x - x_i) + a_2 (x - x_i)^2 + a_3 (x - x_i)^3 ,$$

such that

$$(+) \qquad s_i(x_j) = f_j \quad , \quad i \le j \le i+1 ,$$
$$(++) \qquad s_i''(x_j) = M_j \quad , \quad i \le j \le i+1 .$$

Here, the moments M_i in the nodes are unknown and have to be computed by means of the determining equations.

The coefficients a_k , $0 \le k \le 3$, in $(\diamond)$ can be uniquely determined by the conditions $(+)$ and $(++)$.

Setting $h_i := x_{i+1} - x_i$, we obtain:

$$(\star) \quad s_i(x) = (f_i - \frac{1}{6} h_i^2 M_i) \frac{x_{i+1} - x}{h_i} + (f_{i+1} - \frac{1}{6} h_i^2 M_{i+1}) \frac{x - x_i}{h_i} + M_i \frac{(x_{i+1} - x)^3}{6h_i} + M_{i+1} \frac{(x - x_i)^3}{6h_i} .$$

The condition $s_\Delta \in C^2([a,b])$ gives

$$(\bullet) \quad s'_{i-1}(x_i) \;=\; s'_i(x_i) \quad , \quad 1 \le i \le n-1 \,.$$

Differentiating in

$$(\star) \quad s_i(x) \;=\; (f_i - \frac{1}{6}\, h_i^2\, M_i)\, \frac{x_{i+1}-x}{h_i} + (f_{i+1} - \frac{1}{6}\, h_i^2\, M_{i+1})\, \frac{x-x_i}{h_i} + M_i\, \frac{(x_{i+1}-x)^3}{6h_i} + M_{i+1}\, \frac{(x-x_i)^3}{6h_i} \,.$$

and inserting the result into $(\bullet)$ yields

$$h_{i-1}\, M_{i-1} \;+\; 2\,(h_{i-1}+h_i)\, M_i \;+\; h_i\, M_{i+1} \;=\; 6\,\Big(\frac{f_{i+1}-f_i}{h_i} \;-\; \frac{f_i - f_{i-1}}{h_{i-1}}\Big) \quad , \quad 1 \le i \le n-1 \,.$$

Division by $(h_{i-1}+h_i)$ gives

$$\underbrace{\frac{h_{i-1}}{h_{i-1}+h_i}}_{=:\,\Sigma_i}\, M_{i-1} \;+\; 2\, M_i \;+\; \underbrace{\frac{h_i}{h_{i-1}+h_i}}_{=:\,\Lambda_i}\, M_{i+1} \;=\; \underbrace{\frac{6}{h_{i-1}+h_i}\,\Big(\frac{f_{i+1}-f_i}{h_i} \;-\; \frac{f_i-f_{i-1}}{h_{i-1}}\Big)}_{=:\,D_i} \quad , \quad 1 \le i \le n-1 \,.$$

For the complete (rep. natural, periodic) cubic spline interpolant we obtain the additional boundary conditions:

(i) Complete spline interpolant

$$2\,M_0 + M_1 = \frac{1}{6}\,h_0\,\left(\frac{f_1 - f_0}{h_0} - f_0'\right),$$

$$M_{n-1} + 2\,M_n = \frac{1}{6}\,h_{n-1}\,\left(f_n' - \frac{f_n - f_{n-1}}{h_{n-1}}\right).$$

(ii) Natural spline interpolant

$$M_0 = 0 \quad , \quad M_n = 0\,.$$

(iii) Periodic spline interpolant

$$M_0 = M_n\,,$$

$$\frac{1}{6}\,h_0\,(2\,M_0 + M_1) + \frac{1}{6}\,h_{n-1}\,(M_{n-1} + 2\,M_n) = \frac{f_1 - f_0}{h_0} - \frac{f_n - f_{n-1}}{h_{n-1}}\,.$$

Theorem 4.39 Computation via the moments
Computing the complete (resp. natural, periodic) cubic spline interpolant via the moments $M_1, \ldots, M_{n-1}$ (resp. $M_0, \ldots, M_n$), the moments can be obtained as the solution of a linear algebraic system with a strictly diagonally dominant coefficient matrix.

Remark 4.40 The computation of the cubic spline interpolant
When solving the linear algebraic systems for the slopes m_i resp. the moments M_i, in general we only obtain some approximations $\tilde{m}_i$ resp. $\tilde{M}_i$. The first case results in $\tilde{s}_\Delta \in C^1([a,b])$ with an approximate coincidence of the second derivatives, whereas in the second case we only get $\tilde{s}_\Delta \in C([a,b])$ with an approximate coincidence of the first derivatives.
Therefore, the computation via the slopes is preferable.

4.2.3 Convergence properties

We consider a sequence of grids

$$\Delta_n := \{ a = x_{n,0} < x_{n,1} < \ldots < x_{n,m_n} = b \} ,$$

consisting of $m_n + 1$ mutually disjoint nodes. For given $m_n + 1$ values $f_{n,j}$, $0 \le j \le m_n$, denote by $s_n \in S_{4,\Delta_n}$ the cubic spline interpolant.

Problem: If $f \in C([a,b])$ and $f_{n,j} = f(x_{n,j})$, $0 \le j \le m_n$, does the sequence $(s_n)_{n\in\mathbb{N}}$ converge uniformly on $[a,b]$ to the function f?

Definition 4.41 Minimal, maximal mesh width
The quantities

$$\bar{\Delta}_n := \max_{0\le j\le m_n} | x_{n,j+1} - x_{n,j} | \quad , \quad \underline{\Delta}_n := \min_{0\le j\le m_n} | x_{n,j+1} - x_{n,j} |$$

are called maximal resp. minimal mesh width.

Definition 4.42 Quasiuniformity
A sequence $(\Delta_n)_{n\in\mathbb{N}}$ of grids is said to be quasiuniform, if there exists a constant $\beta > 0$, independent of n, such that

$$\bar{\Delta}_n / \underline{\Delta}_n \le \beta \quad , \quad n \in \mathbb{N} .$$

Theorem 4.43 Uniform convergence of the cubic spline interpolants
Assume that $f \in C([a,b])$ and that $(\ bDelta_n)_{n\in\mathbb{N}}$ is a quasiuniform sequence of grids. Denote by $(s_n)_{n\in\mathbb{N}}$ the associated sequence of complete (resp. natural, periodoc) spline interpolants. Then, there holds:

$$\lim_{n\to\infty} \|f - s_n\|_\infty = 0 \quad \text{für} \quad \bar{\Delta}_n \to 0\,,\ n \to \infty$$

Under additional smoothness properties of the function f, we obtain the following convergence properties:

Theorem 4.44 Uniform convergence of the derivatives
Assume that $f \in C^4([a,b])$ and that $(\Delta_n)_{n\in\mathbb{N}}$ is a quasiuniform sequence of grids. Denote by $(s_n)_{n\in\mathbb{N}}$ the associated sequence of spline interpolants. Then, there exist constants $C_i\ ,\ 0 \le i \le 3$, independent of n and f, such that

$$\|f^{(i)} - s_n^{(i)}\|_\infty \le C_i\, \beta\, \|f^{(4)}\|_\infty\, (\bar{\Delta}_n)^{4-i} \quad , \quad 0 \le i \le 3\,,$$

$$\text{where} \qquad \|f^{(3)} - s_n^{(3)}\|_\infty := \max_{0\le j\le m_n-1}\ \max_{x\in[x_{n,j},x_{n,j+1}]} |f^{(3)}(x) - s_n^{(3)}(x)|\,.$$

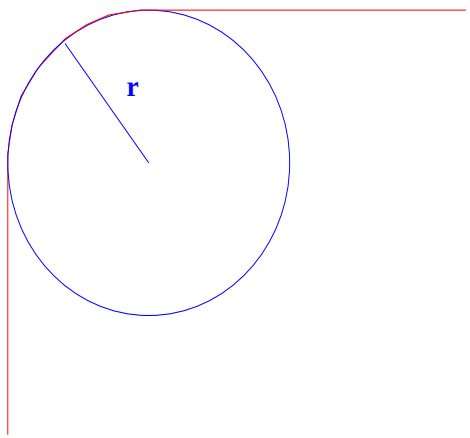
The preceding theorem suggests that even better smoothness properties than $f \in C^4([a,b])$ imply a higher convergence order. This is not the case as shown by the following result:

Theorem 4.44 Maximal order of convergence of the cubic spline interpolant
Assume that $f \in C([a,b])$ and that $(\Delta_n)_{n\in\mathbb{N}}$ is a quaiuniform sequence of grids. Denote by $(s_n)_{n\in\mathbb{N}}$ the associated sequence of complete (natural, periodic) cubic spline interpolants. Moreover, assume that there exists $\mu > 0$ such that

$$\|f - s_n\|_\infty \leq C\,(\bar{\Delta}_n)^{4+\mu}\,.$$

Then, there holds $f^{(4)}(x) = 0\ ,\ x \in [a,b]$, i.e., $f \in P_3([a,b])$.

4.2.4 Geometric properties of cubic spline interpolants



Parameterized curve:

$y : [a, b] \to \mathbb{R} \quad , \quad y \in C^2([a, b])$

Curvature of the curve in $x \in [a, b]$:

$$\kappa(x) \;=\; \frac{y''(x)}{(1 + y'(x)^2)^{3/2}} \quad , \quad |\kappa(x)| \;=\; 1/r$$

r: radius of the supporting circle

For "small" $y'(x)$, i.e., $|y'(x)| \ll 1$ there holds:

$$\frac{y''(x)}{(1 + y'(x)^2)^{3/2}} \;\approx\; y''(x)$$

A suitable measure for the curvature of the curve is the L^2-norm of the second derivative, i.e.:

$$\|y''\|_{0,2} \;:=\; \Big(\int_a^b (y''(x))^2 \, dx\Big)^{1/2} .$$

Theorem 4.46 Geometric property

Assume that (x_i, f_i) , $0 \le i \le n$, are $n+1$ pairs satisfying $\Delta := \{a = x_0 < x_1 < \ldots < x_n = b\}$. Moreover, let $s_\Delta \in S_{4,\Delta}$ be a cubic spline interpolant and $y \in C^2([a,b])$ an arbitrary interpolating function such that

$$[\, s''_\Delta(x)\,(y'(x) \;-\; s'_\Delta(x))\,]\Big|_a^b \;=\; 0\,.$$

Then, there holds:

$$\|s''_\Delta\|_{0,2} \;\le\; \|y''\|_{0,2}\,.$$

Proof: Setting $y'' = s''_\Delta + (y'' - s''_\Delta)$, we have:

$$(\star)\quad \int_a^b (y'')^2\,dx \;=\; \int_a^b (s''_\Delta)^2\,dx \;+\; 2\int_a^b s''_\Delta\,(y'' - s''_\Delta)\,dx \;+\; \int_a^b (y'' - s''_\Delta)^2\,dx\,.$$

For the second term on the right-hand side in $(\star)$, partial integration yields:

$$\int_a^b s''_\Delta\,(y'' - s''_\Delta)\,dx \;=\; \sum_{i=0}^{n-1}\int_{x_i}^{x_{i+1}} s''_\Delta\,(y'' - s''_\Delta)\,dx \;=\; \sum_{i=0}^{n-1}\Big\{\,[s''_\Delta\,(y' - s'_\Delta)]\Big|_{x_i}^{x_{i+1}} \;-\; \int_{x_i}^{x_{i+1}} \underbrace{s'''_\Delta}_{=\text{const.}}\,(y' - s'_\Delta)\,dx\,\Big\}$$

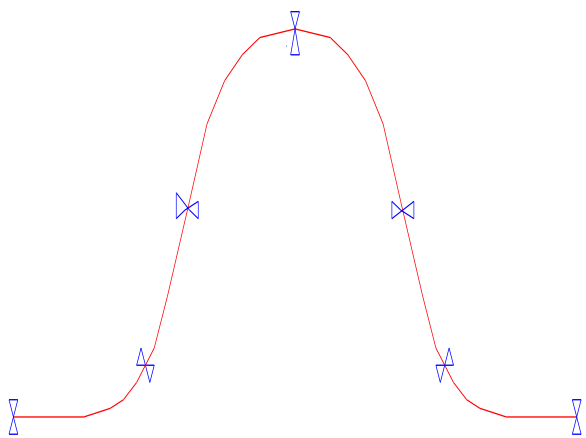
$$= \sum_{i=0}^{n-1}\Big\{\,[s''_\Delta\,(y' - s'_\Delta)]\Big|_{x_i}^{x_{i+1}} \;-\; s'''_\Delta\,(y' - s'_\Delta)\Big|_{x_i}^{x_{i+1}}\,\Big\} \;=\; [\,s''_\Delta\,(y' - s'_\Delta)\,]\Big|_a^b \;=\; 0\,.$$

Corollary 4.47 Geometric property

Assume that $s_\Delta \in S_{4,\Delta}$ is the complete (resp. natural, periodic) cubic spline interpolating in (x_i, f_i) , $0 \le i \le n$. Then, for any interpolating function $y \in C^2([a,b])$, satisfying the same boundary conditions, there holds:

$$\|s''_\Delta\|_{0,2} \le \|y''\|_{0,2} \, .$$

Remark 4.48 Spline



Flexible wooden ruler used as a drawing device fixed in the nodes $x_i, 0 \le i \le n$.

Stationary state:

State for which the "bending energy" is minimal:

$$E = \int_a^b \left(\frac{y''(x)}{(1 + y'(x)^2)^{3/2}} \right)^2 dx$$

The cubic spline interpolant approximately describes the stationary state of the drawing device

4.2.5 B-splines

Construction of a basis of $S_{m,\Delta}$, $\Delta := \{a = x_0 < x_1 < \ldots < x_n = b\}$, $m \in \mathbb{N}$, such that

(•) Evaluation with respect to the basis representation is well conditioned,

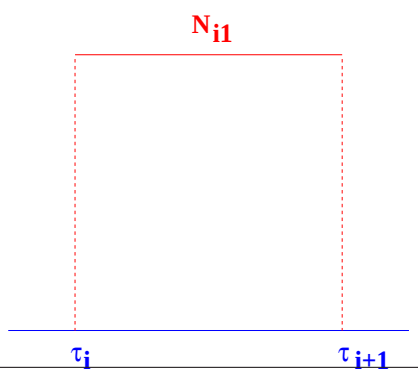
(•) Basis functions have minimal support in $[a, b]$.

Definition 4.49 B-splines

Assume that $\tau_1 \leq \tau_2 \leq \ldots \leq \tau_n$ is an arbitrary sequence of nodes and $f(t) := (t - x)_+^{m-1}$, $m \in \mathbb{N}$, $x \in \mathbb{R}$. If $\tau_i < \tau_{i+m}$, $1 \leq i \leq n - m$, the B-splines $N_{im}, 1 \leq i \leq n - m$, of order $m, 1 \leq m \leq n$, are defined as follows:

$$(*) \quad N_{im}(x) := (\tau_{i+m} - \tau_i)\, f[\tau_i, \ldots, \tau_{i+m}] \,.$$

Remark 4.50 B-splines in case $m = 1$



For $m = 1$ it follows that:

$$N_{i1}(x) = (\tau_{i+1} - \tau_i)\, f[\tau_i, \tau_{i+1}] = (\tau_{i+1} - \tau_i)\, \frac{(\tau_{i+1}-x)_+^0 - (\tau_i - x)_+^0}{\tau_{i+1} - \tau_i} =$$

$$= \begin{cases} 1 & , \quad \tau_i \leq x \leq \tau_{i+1} \\ 0 & , \quad \text{else} \end{cases}$$

The B-splines satisfy a recursion which can be deduced from $(*)$ by using the Leibniz-rule for divided differences:

Lemma 4.51 Leibniz-rule for divided differences
Assume $g, h \in C^n([a,b])$ and $x_i \in [a,b]$, $0 \le i \le n$. Then, there holds:

$$(gh)[x_0, ..., x_n] = \sum_{i=0}^{n} g[x_0, ..., x_i]\, h[x_i, ..., x_n] .$$

Theorem 4.52 Recursion for B-splines

For $m > 1$, the B-splines N_{im} satisfy the recursion

$$(\star) \quad N_{im}(x) = \frac{x - \tau_i}{\tau_{i+m-1} - \tau_i} N_{i,m-1}(x) + \frac{\tau_{i+m} - x}{\tau_{i+m} - \tau_{i+1}} N_{i+1,m-1}(x) .$$

Here, it has to be observed that in case of multiple nodes the corresponding terms on the right-hand side in $(\star)$ are omitted according to the convention " $0/0 = 0$ ".

Proof: Setting $g(t) := (t-x)$ and $h(t) := (t-x)_+^{m-2}$ and using the Leibniz-rule, we obtain

$$N_{im}(x) \;=\; (\tau_{i+m}-\tau_i)\,(gh)[\tau_i,\ldots,\tau_{i+m}] \;=\; (\tau_{i+m}-\tau_i)\sum_{k=0}^{m} g[\tau_i,\ldots,\tau_{i+k}]\,h[\tau_{i+k},\ldots,\tau_{i+m}]\,.$$

In view of the linearity of g, all divided differences of higher than first order disappear:

$$\begin{aligned}
N_{im}(x) \;&=\; (\tau_{i+m}-\tau_i)\,\{\,g[\tau_i]\,h[\tau_i,\ldots,\tau_{i+m}] \;+\; \underbrace{g[\tau_i,\tau_{i+1}]}_{=1}\,h[\tau_{i+1},\ldots,\tau_{i+m}]\,\} \;=\\
&=\; (\tau_{i+m}-\tau_i)\,\{\,(\tau_i-x)\,\frac{h[\tau_{i+1},\ldots,\tau_{i+m}]-h[\tau_i,\ldots,\tau_{i+m-1}]}{\tau_{i+m}\;-\;\tau_i} \;+\; h[\tau_{i+1},\ldots,\tau_{i+m}]\,\} \;=\\
&=\; \frac{x-\tau_i}{\tau_{i+m-1}-\tau_i}\,(\tau_{i+m-1}-\tau_i)\,h[\tau_i,\ldots,\tau_{i+m-1}] \;+\; \frac{\tau_{i+m}-x}{\tau_{i+m}-\tau_{i+1}}\,(\tau_{i+m}-\tau_{i+1})\,h[\tau_{i+1},\ldots,\tau_{i+m}] \;=\\
&=\; \frac{x-\tau_i}{\tau_{i+m-1}-\tau_i}\,N_{i,m-1}(x) \;+\; \frac{\tau_{i+m}-x}{\tau_{i+m}-\tau_{i+1}}\,N_{i+1,m-1}(x)\,.
\end{aligned}$$

Lemma 4.53 Properties of B-splines

The B-splines N_{im} , $1 \le i \le n-m$, as defined by $(*)$ have the following properties:

$$\begin{aligned} &(B1) \qquad \operatorname{supp} N_{im} \subset [\tau_i, \tau_{i+m}] \,, \\ &(B2) \qquad N_{im}(x) \ge 0 \,, \; x \in \mathbb{R} \,, \\ &(B3) \qquad N_{im}|_{[\tau_j, \tau_{j+1}]} \in P_{m-1}([\tau_j, \tau_{j+1}]) \,. \end{aligned}$$

Corollary 4.54 Multiple nodes

Assume that τ_i is a k-fold node, i.e.,

$$\tau_{i-1} \;<\; \tau_i \;=\; \tau_{i+1} \;=\; \ldots \;=\; \tau_{i+k-1} \;<\; \tau_{i+k} \,.$$

Then, N_{im} is at least $(m-k-1)$ times continuously differentiable in $x = \tau_i$ and

$$N'_{im}(x) \;=\; (m-1)\left[\, \frac{N_{i,m-1}(x)}{\tau_{i+m-1} - \tau_i} \;-\; \frac{N_{i+1,m-1}(x)}{\tau_{i+m} - \tau_{i+1}} \,\right] .$$

Construction of a basis of $S_{m,\Delta}$

For the grid $\Delta := \{a = x_0 < \ldots < x_n = b\}$ we define extended nodes $T = \{\tau_i\}_{i=1}^{2m+n-1}$ with m-fold nodes in $a = x_0$ and $b = x_n$ according to

$$\begin{array}{lcccccccccccccc} \Delta & : & & & a & = & x_0 & < & x_1 & < & \ldots & < & x_n & = & b \\ T & : & \tau_1 & = & \ldots & = & \tau_m & < & \tau_{m+1} & < & \ldots & < & \tau_{m+n} & = & \ldots & = & \tau_{2m+n-1} \end{array}$$

In view of (B3) we have $N_{im}|_{[x_j,x_{j+1}]} \in P_{m-1}([x_j, x_{j+1}])$, $0 \le j \le n-1$, and Corollary 4.54 implies that N_{im} is $(m-2)$ times continuously differentiable in $x = x_j$, $1 \le j \le n-1$,. Hence, we get:

$$N_{im} \in S_{m,\Delta} \quad , \quad 1 \le i \le n+m-1 \, .$$

The linear independence of the N_{im} remains to be shown.

Lemma 4.55 Marsden identity

Let $T = \{\tau_i\}_{i=1}^{n+2m-1}$ be the extended sequence of nodes and for $s \in \mathbb{R}$ assume that

$$\varphi_{im}(s) := \prod_{j=1}^{m-1} (\tau_{i+j} - s) .$$

Then, for all $x \in [a, b]$ the so-called Marsden identity holds true:

$$(\diamond) \quad (x - s)^{m-1} = \sum_{i=1}^{n+m-1} \varphi_{im}(s)\, N_{im}(x) .$$

Proof: Induction on m.

Corollary 4.56 Partition of unity

The B-splines N_{im} , $1 \le i \le n + m - 1$, form a partition of unity on $[a, b]$ according to

$$(P1) \qquad P_{m-1}([a, b]) \subset \text{span}\,\{N_{1m}, \ldots, N_{n+m-1,m}\} ,$$

$$(P2) \qquad \sum_{i=1}^{n+m-1} N_{im}(x) = 1 \quad , \quad x \in [a, b] .$$

Proof: Taking into account the Marsden identity

$$(\diamond) \quad (x - s)^{m-1} = \sum_{i=1}^{n+m-1} \varphi_{im}(s)\, N_{im}(x)$$

for the ℓ-th derivative of the function $f(s) := (x-s)^{m-1}$ it follows that

$$f^{(\ell)}(0) = (m-1)(m-2)\ldots(m-\ell)\,(-1)^{\ell}\, x^{m-\ell-1} = \sum_{i=1}^{n+m-1} \varphi_{im}^{(\ell)}(0)\, N_{im}(x)\,.$$

In particular, for $m-\ell-1 = k$ we obtain:

$$x^k = \frac{(-1)^{m-k-1}}{(m-1)\ldots(k+1)} \sum_{i=1}^{n+m-1} \varphi_{im}^{(m-k-1)}(0)\, N_{im}(x)\,.$$

For $k = 0$, the assertion (P2) follows in view of

$$\varphi_{im}^{(m-1)}(s) = \Big(\prod_{j=1}^{m-1}(\tau_{i+j} - s)\Big)^{(m-1)} = \big[\,(-1)^{m-1}\, s^{m-1} + \ldots\,\big]^{(m-1)} = (-1)^{m-1}\,(m-1)!\,.$$

Theorem 4.57 Linear independence of the B-splines

The B-splines N_{im} , $1 \le i \le n+m-1$, are linear independent, i.e.,

$$\sum_{i=1}^{n+m-1} c_i \, N_{im}(x) \;=\; 0 \quad , \quad x \in [a,b] \quad \Longrightarrow \quad c_i = 0 \; , \; 1 \le i \le n+m-1 \; .$$

Proof: Consider $[x_j, x_{j+1}] = [\tau_{m+j}, \tau_{m+j+1}]$, $0 \le j \le n-1$. Since $N_{im}|_{[\tau_{m+j},\tau_{m+j+1}]} = 0$, $i \notin \{j+1, \ldots, j+m\}$, it follows that

$$(+) \quad \sum_{i=j+1}^{j+m} c_i \, N_{im}(x) \;=\; 0 \quad , \quad x \in [x_j, x_{j+1}] \; .$$

In view of (P1), $p_0 \in P_{m-1}([a,b])$, $p_0 \equiv 0$, can be written as a linear combination of the B-splines $N_{im}, 1 \le i \le n+m-1$,. Since $p_0|_{[x_j,x_{j+1}]}$ is uniquely determined by $m = \dim P_{m-1}([a,b])$ values, taking into account $(+)$, the N_{im} , $j+1 \le i \le j+m$, must be linearly independent, i.e., we must have $c_i = 0$, $j+1 \le i \le j+m$.

Korollar 4.58 de Boor points

Every spline $s_\Delta \in S_{m,\Delta}$ admits the unique representation

$$s_\Delta = \sum_{i=1}^{n+m-1} d_i \, N_{im} \, .$$

The coefficients d_i , $1 \le i \le n+m-1$, are called the de Boor points of s_Δ.

Theorem 4.59 Recursive evaluation in the B-spline representation

Let $s_\Delta \in S_{m,\Delta}$ and $x \in [a,b]$. Then, $s_\Delta(x)$ can be recursively computed according to

$$s_\Delta(x) = \sum_{i=j+1}^{n+m-1} d_i^j(x) \, N_{i,m-j}(x) \, ,$$

where $d_i^0(x) := d_i$ and for $j > 0$ there holds:

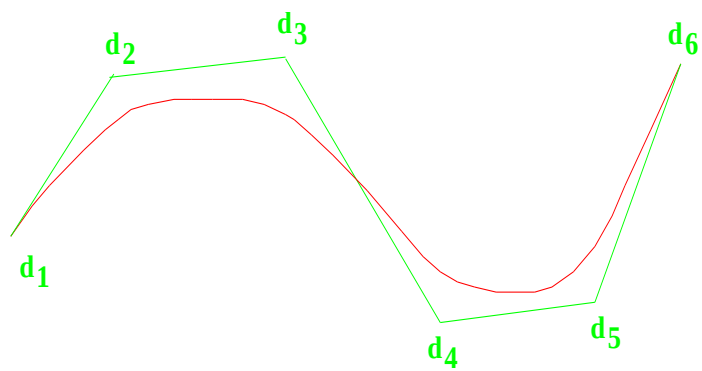
$$d_i^j(x) := \begin{cases} \frac{x - \tau_i}{\tau_{i+m-j} - \tau_i} \, d_i^{j-1}(x) + \frac{\tau_{i+m-j} - x}{\tau_{i+m-j} - \tau_i} \, d_{i-1}^{j-1}(x) & , \quad \tau_{i+m-j} \neq \tau_i \\ 0 & , \quad \text{else} \end{cases} \, .$$

In particular, we have $s(x) = d_i^{m-1}(x)$, $x \in [\tau_i, \tau_{i+1}]$.

Proof: Induction argument.

Remark 4.60 de Boor algorithm

The value $s_\Delta(x) = d_i^{m-1}(x)$ is obtained by a successive convex combination of the de Boor points according to the following scheme



$$
\begin{array}{cccc}
d_{i-m+1}^0(x) & & & \\
 & d_{i-m+2}^1(x) & & \\
d_{i-m+2}^0(x) & & \ddots & \\
\vdots & & & d_i^{m-1}(x) \\
d_{i-1}^0(x) & & & \\
 & d_i^1(x) & & \\
d_i^0(x) & & &
\end{array}
$$

Theorem 4.61 Condition of the evaluation in the B-spline representation

There exists a constant $D_m > 0$, depending only on the order m, such that

$$(\diamond) \quad D_m \max_{1 \le i \le n+m-1} |d_i| \le \| \sum_{i=1}^{n+m-1} d_i N_{im} \|_\infty \le \max_{1 \le i \le n+m-1} |d_i| .$$

Remark 4.62 About the condition of the evaluation in the B-spline representation

According to $(\diamond)$, perturbations in $s_\Delta(x) = \sum_i d_i N_{im}(x)$ of a spline $s_\Delta \in S_{m,\Delta}$ as well as in the coefficients d_i can be mutually estimated. Therefore, the evaluation in the B-spline representation is well conditioned.

For this reason, the basis N_{im} , $1 \le i \le n+m-1$, is referred to as a well conditioned basis.

4.3 Trigonometric interpolation

4.3.1 Discrete Fourier transformation

Reminder from calculus: Fourier transformation and Fourier series

The Fourier transform of a 2π-periodic function $f : \mathbb{R} \to C$ is a function $\hat{f} := Ff : Z \to C$ with

$$(*) \quad \hat{f}(k) = \frac{1}{2\pi} \int_0^{2\pi} f(t) \exp(-ikt)\, dt \quad , \quad k \in Z \, .$$

The inverse mapping $f = F^{-1}\hat{f}$ is given by the Fourier series

$$(**) \quad f(t) = \sum_{k=-\infty}^{+\infty} \hat{f}(k) \exp(ikt) \, .$$

Choosing an equidistant partition of $[0, 2\pi)$ with respect to the nodes $t_j := 2\pi j/n, 0 \le j \le n-1$ $n \in \mathbb{N}$, and using the trapezoidal rule

$$\int_0^{2\pi} g(t)\, dt \doteq \frac{2\pi}{n} \sum_{j=0}^{n-1} g(t_j) \, ,$$

$(*)$ yields:

$$\hat{f}(k) \doteq \frac{1}{n} \sum_{j=0}^{n-1} f(t_j) \exp(-2\pi ijk/n) \, .$$

Definition 4.63 Discrete Fourier transformation
Assume that $f : \mathbb{R} \to \mathbb{C}$ is 2π-periodic, $t_j := 2\pi j/n$, $0 \le j \le n-1$, $n \in \mathbb{N}$, and $f_j := f(t_j)$, $0 \le j \le n-1$. Then, the function

$$\mathcal{F}_n : \mathbb{C}^n \to \mathbb{C}^n \quad , \quad (f_j) \longmapsto (c_j)$$

$$c_k = \frac{1}{n} \sum_{j=0}^{n-1} f_j \exp(-2\pi i j k/n) \quad , \quad 0 \le k \le n-1$$

is called the discrete Fourier transformation. The inverse mapping $\mathcal{F}_n^{-1}$ is given by

$$(\star) \quad f_j = \sum_{k=0}^{n-1} c_k \exp(2\pi i j k/n) \quad , \quad 0 \le j \le n-1 \, .$$

Remark 4.64 Truncated Fourier series

If $n = 2m+1$, then setting $c_k = c_{m+k}$, $-m \le k < 0$, we obtain:

$$f_j = \sum_{k=-m}^{+m} c_k \exp(2\pi i j k/n) \, .$$

Hence, $(\star)$ can be interpreted as a truncated Fourier series.

Definition 4.65 Trigonometric polynomial
Let $c_k \in \mathbb{C}$, $0 \le k \le n-1$, $n \in \mathbb{N}$. Then

$$T_{n-1}(t) \;=\; \sum_{k=0}^{n-1} c_k \exp(ikt) \;,\; t \in [0, 2\pi)$$

is called a trigonometric polynomial.

Definition 4.66 Trigonometric interpolation problem
Assume that $f : \mathbb{R} \to \mathbb{C}$ is 2π-periodic, $t_j := 2\pi j/n$, $0 \le j \le n-1$, $n \in \mathbb{N}$, and $f_j = f(t_j)$, $0 \le j \le n-1$. The computation of a trigonometric polynomial T_{n-1} such that

$$(\diamond) \quad T_{n-1}(t_j) \;=\; f_j \quad , \quad 0 \le j \le n-1 \;,$$

is called a trigonometric interpolation problem.

Theorem 4.57 Solution of the trigonometric interpolation problem
The trigonometric interpolation problem $(\diamond)$ is uniquely solvable. The coefficients c_k, $0 \le k \le n-1$, are given by

$$c_k \;=\; \frac{1}{n} \sum_{j=0}^{n-1} f_j \exp(-2\pi ijk/n)$$

Proof: Denoting by

$$\omega_j \;=\; \exp(2\pi ij/n) \quad , \quad 0 \le j \le n-1$$

the n-th unit roots, $(\diamond)$ is equivalent to the complex polynomial interpolation problem:
Compute $T_{n-1}(z) = \sum_{k=0}^{n-1} c_k z^k$, $z = \exp(it)$ such that

$$T_{n-1}(\omega_j) \;=\; f_j \quad , \quad 0 \le j \le n-1 \; .$$

Since $\omega_j \neq \omega_\ell$ for $0 \le j \neq \ell \le n-1$, the interpolation problem is uniquely solvable.
The representation of the coefficients according to

$$c_k \;=\; \frac{1}{n} \sum_{j=0}^{n-1} f_j \exp(-2\pi ijk/n)$$

follows from the orthogonality property

$$\frac{1}{n} \sum_{j=0}^{n-1} \omega_j^k \, \omega_j^{-\ell} \;=\; \delta_{k\ell} \; .$$

4.3.2 Fast Fourier Transform (FFT)

The Fast Fourier Transform is the efficient implementation of the discrete Fourier transform

$$(*) \quad c_k = \frac{1}{n} \sum_{j=0}^{n-1} f_j \exp(-2\pi ijk/n) \quad , \quad 0 \le k \le n-1$$

resp. of the inverse mapping

$$(**) \quad f_j = \sum_{k=0}^{n-1} c_k \exp(2\pi ijk/n) \quad , \quad 0 \le j \le n-1$$

in case $n = 2^p$, $p \in \mathbb{N}$.

Explanation of the FFT for $(**)$:

Computational work in case of a "naive" approach:

$O(n^2)$ complex additions, multiplications, evaluation of trigonometric functions

Idea for a reduction of the computational work: Let $n = 2m$, $m = 2^{p-1}$. Partition $(**)$

(•) w.r.t. k into even $(2k)$ and odd $(2k+1)$ terms,

(•) w.r.t. j into the lower half $(0, ..., m-1)$ and into the upper half $(m, ..., 2m-1)$.

For $0 \le j \le m-1$, the partition leads to:

$$(+) \quad f_j = \sum_{k=0}^{m-1} c_{2k} \exp(2\pi ijk/m) + \exp(\pi ij/m) \sum_{k=0}^{m-1} c_{2k+1} \exp(2\pi ijk/m) ,$$

$$(++) \quad f_{j+m} = \sum_{k=0}^{m-1} c_{2k} \exp(2\pi ijk/m) - \exp(\pi ij/m) \sum_{k=0}^{m-1} c_{2k+1} \exp(2\pi ijk/m) .$$

Since both partial sums in $(+)$ also appear in $(++)$:
Reduction of the computational work by 50 %, but additionally m additions resp. subtractions, multiplications and evaluations of trigonometric functions ("working costs").
Recursive application of the splitting yields the following partition consisting of p steps

							f_j									$0 \le j \le 2^p - 1$
1.step:			f_j								$f_{j+2^{p-1}}$					$0 \le j \le 2^{p-1} - 1$
2.step:		f_j		$f_{j+2^{p-2}}$				$f_{j+2^{p-1}}$				$f_{j+2^{p-1}+2^{p-2}}$				$0 \le j \le 2^{p-2} - 1$
		.	.	.	.	.	.	.	.	.	.	.	.	.		
		.	.	.	.	.	.	.	.	.	.	.	.	.		
		.	.	.	.	.	.	.	.	.	.	.	.	.		
p-th step:	f_0	f_1	.	.	.	.	.	.	.	.	.	.	.	f_{2^n-2}	f_{2^n-1}	

Computational work (only sum of the working costs):

$p\ n/2$ additions, subtractions and multiplications

$\frac{n}{2} + \frac{n}{4} + \ldots + \frac{n}{2^p} = n \sum_{i=1}^{p} 2^{-i} \approx 2n$ trigonometric functions

Since $n = 2^p$ and hence, $p = \log_2 n \implies O(n \log_2 n)$ essential operations

Practical implementation of the FFT:

Input: $c_k\ ,\ 0 \le k \le n-1$ in array $c[0, \ldots, n-1]$

Output: $f_j\ ,\ 0 \le j \le n-1$ in the same array, i.e., the c-array is overwritten

Realization: Sorting and combination steps

1. Sorting step:

Sort all components $c_k\ ,\ 0 \le k \le n-1$, according to the principle "even" before "odd":

$$\boxed{0 \mid 1 \mid 2 \mid \cdots \mid 2^p - 1} \longmapsto \boxed{0 \mid 2 \mid \cdots \mid 2^p - 2 \mid 1 \mid 3 \mid \cdots \mid 2^p - 1}$$

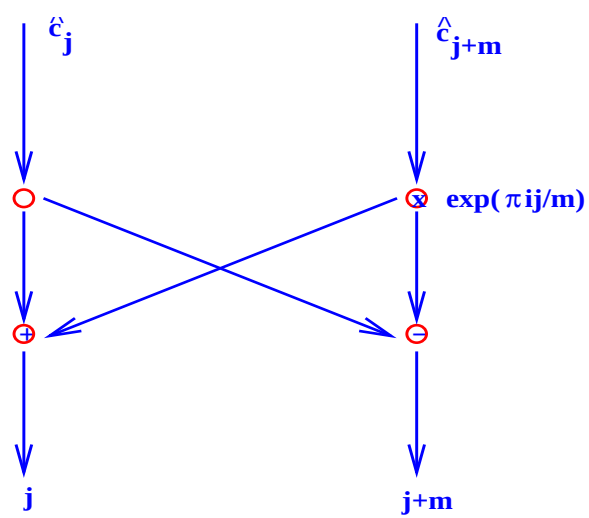
p-th combination step:

$$f_j = \hat{c}_j + \exp(\pi i j/m)\,\hat{c}_{j+m} \quad , \quad f_{j+m} = \hat{c}_j - \exp(\pi i j/m)\,\hat{c}_{j+m} \quad , \quad 0 \le j \le m-1 \,,$$

where

$$\hat{c}_j = \sum_{k=0}^{m-1} c_k \exp(2\pi i j k/m) \quad , \quad \hat{c}_{j+m} = \sum_{k=0}^{m-1} c_{k+m} \exp(2\pi i j k/m) \quad , \quad 0 \le j \le m-1 \,.$$

f_j and f_{j+m} are stored in $\hat{c}_j$ and $\hat{c}_{j+m}$.



"butterfly diagram"