Department of Mathematics
University of Houston
Numerical Analysis I
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Numerical Analysis I
(2nd Homework Assignment)

Exercise 5 (*Block Gauss elimination*)

Let $A \in \mathbf{R}^{N \times N}$, $N := \sum_{i=1}^{m} n_i$, $n_i \in \mathbf{N}$, $1 \le i \le m$ and $b \in \mathbf{R}^N$ be block structured according to

$$A = \begin{pmatrix} A_{11} & \cdot & \cdot & \cdot & A_{1m} \\ A_{21} & \cdot & \cdot & \cdot & A_{2m} \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ A_{m1} & \cdot & \cdot & \cdot & A_{mm} \end{pmatrix} \quad , \quad b = (b_1, \cdot, \cdot, \cdot, b_m)^T ,$$

where $A_{ij} \in \mathbf{R}^{n_i \times n_j}$, $1 \le i,j \le m$, $b_i \in \mathbf{R}^{n_i}$, $1 \le i \le m$.

(i) Consider the solution of the linear algebraic system

$$Ax = b .$$

Use a corresponding structuring of the solution vector $x \in \mathbf{R}^N$ and give a block variant of Gauss elimination.

(ii) Give a block variant of the LR-decomposition for block tridiagonal matrices $A \in \mathbf{R}^{N \times N}$, i.e., $A_{ij} = 0$ for $| i - j | \ge 2$, $1 \le i,j \le m$.

Exercise 6 (*Schur complement*)

Let $A \in \mathbf{R}^{N \times N}$, $N := n_1 + n_2$, $n_i \in \mathbf{N}$, $1 \le i \le 2$ be a symmetric positive definite block matrix of the form

$$A = \begin{pmatrix} A_{11} & A_{12} \\ A_{12}^T & A_{22} \end{pmatrix} .$$

Prove that the Schur complement $S = A_{22} - A_{12}^T A_{11}^{-1} A_{12}$ is also symmetric positive definite.

Exercise 7 (*Positive definite symmetric Toeplitz matrices*)

A matrix $A \in \mathbf{R}^{n \times n}$ is called a normalized symmetric Toeplitz matrix, if

$$a_{ij} = r_{|i-j|} \ , \ 1 \le i,j \le n ,$$

where $r_0 = 1$ and $r = (r_1, ..., r_n)^T \in \mathbf{R}^n$ such that A is positive definite. The solution of the linear algebraic system

$$Ax = -r$$

is referred to as the Yule-Walker problem which plays a significant role in algorithms for the reconstruction of noisy signals.

Consider a partitioning of the matrix A according to

$$A = \begin{pmatrix} \tilde{A} & P_{n-1}\tilde{r} \\ \tilde{r}^T P_{n-1} & 1 \end{pmatrix} ,$$

where $\tilde{A} \in \mathbf{R}^{(n-1)\times(n-1)}$, $\tilde{r} = (r_1, ..., r_{n-1})^T$ and

$$P_{n-1} = \begin{pmatrix} 0 & 0 & 0 & \cdot & \cdot & 0 & 0 & 1 \\ 0 & 0 & 0 & \cdot & \cdot & 0 & 1 & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ 0 & 1 & 0 & \cdot & \cdot & 0 & 0 & 0 \\ 1 & 0 & 0 & \cdot & \cdot & 0 & 0 & 0 \end{pmatrix} .$$

Use a corresponding partitioning of the right-hand side and of the solution vector. Develop a recursive algorithm for the Yule-Walker problem which computes the solution for the dimension n, provided the solution for the dimension $n-1$ is known.

Exercise 8 (*Cholesky decomposition of positive semidefinite matrices*)

Assume that $A \in \mathbf{R}^{n\times n}$ is positive semidefinite of rank $r < n$. Prove the following two assertions:

(i) There exists an upper triangular matrix R with nonnegative diagonal elements such that

$$A = R^T R .$$

(ii) There exists a permutation matrix P such that $P^T AP$ has a unique Cholesky decomposition of the form

$$P^T AP = R^T R \quad , \quad R = \begin{pmatrix} R_{11} & R_{12} \\ 0 & 0 \end{pmatrix} ,$$

where R_{11} is an $r \times r$ upper triangular matrix with positive diagonal elements.

Delivery of the homework at latest on September 11, 2009. The homework may be submitted either electronically (rohop@math.uh.edu) or as a hardcopy in class.