Department of Mathematics
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Numerical Analysis I
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Numerical Analysis I
(3rd Homework Assignment)

Exercise 9 (*Convergence of the Jacobi iteration)*)

A matrix $A \in \mathbf{R}^{n\times n}$ is called strongly diagonally dominant, if

$$| \, a_{ii} \, | \; > \; \sum_{j\neq i} | \, a_{ij} \, | \; , \;\; 1 \leq i \leq n \; .$$

Prove the convergence of the Jacobi iteration applied to the linear algebraic system $Ax = b$, $b \in \mathbf{R}^n$ in case of strongly diagonally dominant matrices.

Hint: Use Gershgorin's theorem: For any matrix $A \in \mathbf{C}^{n\times n}, n \in \mathbf{N}$ there holds

$$\sigma(A) \;\; \subset \;\; \bigcup_{i=1}^{n} \{ \lambda \in \mathbf{C} \; | \; | \, \lambda - a_{ii} \, | \leq \sum_{j\neq i} | \, a_{ij} \, | \} \; .$$

Exercise 10 (*Scaling invariance of the Gauss Seidel iteration*)

Scaling means the multiplication of quantities of interest by constant scalars. In physical and technical applications, this typically means the transformation of units.

A numerical method for a given problem is said to be scaling invariant, if its application to the scaled problem gives rise to the same results as in the unscaled case.

Show that the Gauss Seidel iteration applied to the linear algebraic system $Ax = b$, $A \in \mathbf{R}^{n\times n}$, $b \in \mathbf{R}^n$, is scaling invariant. Observe that in this case the scaled problem $\hat{A}\hat{x} = \hat{b}$ is given by $\hat{A} := D_S A D_S$, $\hat{x} := D_S^{-1} x$, $\hat{b} := D_S b$, where D_S stands for a regular diagonal matrix.

Exercise 11 (*Symmetric Gauss Seidel iteration*)

Consider the linear algebraic system $Ax = b$ with regular $A \in \mathbf{R}^{n\times n}$ and nonzero diagonal elements. An iteration step of the symmetric Gauss Seidel iteration consists of two substeps: Firstly, implement one step of the ordinary Gauss Seidel iteration and secondly, apply another Gauss Seidel step, but with the components of the solution vector in reversed order.
With regard to the decomposition $A = D - E - F$ of A into the diagonal part D, the lower triangular part $-E$ and the upper triangular part $-F$ construct the iteration matrix M of the symmetric Gauss Seidel iteration.

Exercise 12 (*Convergence of the SOR iteration*)

Assume that $A \in \mathbf{R}^{n\times n}$ is symmetric positive definite and let $A = D - E - E^T$ be the decomposition of A into the diagonal part D and the lower triangular resp. upper triangular parts $-E$ and $-E^T$. For $\omega \in (0, 2)$, denote by

$$M_{SOR}(\omega) \;:=\; (D - \omega E)^{-1}\{(1-\omega)D + \omega E^T\}$$

the iteration matrix of the SOR iteration.

(i) Show that for $\hat{M} := A^{\frac{1}{2}} M_{SOR}(\omega) A^{-\frac{1}{2}}$ there holds:

$$\hat{M}^T \hat{M} \;=\; I \;-\; (\frac{2}{\omega} - 1)(ZZ^T)^{-1} \;,$$

where $Z := A^{-\frac{1}{2}} W_{SOR}(\omega) D^{-\frac{1}{2}}$ and $W_{SOR}(\omega) := \frac{1}{\omega} D - E$.
[Hint: Use $A - W_{SOR}(\omega) - W_{SOR}^T(\omega) = (1 - \frac{2}{\omega})D$.]

(ii) Taking advantage of part (i) of this exercise, show that:

$$\|M_{SOR}(\omega)\|_A \;=\; \sqrt{1 - (\frac{2}{\omega} - 1)/\|Z\|_2^2} \quad .$$

(iii) Prove that

$$\|Z\|_2^2 \;\leq\; \frac{1}{c}$$

holds true if and only if

$$W_{SOR}(\omega) D^{-1} W_{SOR}^T(\omega) \;\leq\; \frac{1}{c} A \quad .$$

Delivery of the homework at latest on September 25, 2005. The homework may be submitted either electronically (rohop@math.uh.edu) or as a hardcopy in class.