Department of Mathematics
University of Houston
Numerical Analysis I
Dr. Ronald H.W. Hoppe

Numerical Analysis I
(4th Homework Assignment)

Exercise 13 (*Gradient method and semi-iterative Richardson iteration)*)

Let $A \in \mathbf{R}^{n \times n}$ be symmetric positive definite with the extreme eigenvalues $\lambda := \lambda_{min}(A)\,,\, \Lambda := \lambda_{max}(A)$ and let $\kappa := \Lambda/\lambda$ be the spectral condition of A. Further, let $b \in \mathbf{R}^n$ and $x^0 \in \mathbf{R}^n$.

(i) Consider the semi-iterative Richardson iteration

$$y^{m+1} \;=\; y^m \;+\; \Theta_{m+1}\,(b \;-\; Ay^m) \;\;,\;\; m \geq 0 \;\;.$$

Specify the values of $y^0 \in \mathbf{R}^n$ and Θ_{m+1} for which the sequence of iterates $(y^m)_{\mathbf{N}}$ corresponds to the sequence $(x^m)_{\mathbf{N}}$ obtained by the gradient method.

(ii) Show that for any initial vector $x^0 \in \mathbf{R}^n$ the sequence $(x^m)_{\mathbf{N}}$, obtained by the gradient method, converges to $x^* := A^{-1}b$. Moreover, verify the estimates

$$F(x^m) \;-\; F(x^*) \;\;\leq\;\; (\frac{\kappa-1}{\kappa+1})^{2m}\,[\,F(x^0) \;-\; F(x^*)\,] \;\;,$$

$$\|x^m \;-\; x^*\|_A \;\;\leq\;\; (\frac{\kappa-1}{\kappa+1})^{m}\,\|x^0 \;-\; x^*\|_A \;\;,$$

where $F(x) := \frac{1}{2} < Ax, x > \;-\; < b, x >$.

[Hint: Utilize Theorem 1.9 as presented in class and take advantage of the fact that x^m minimizes the error $\|x^m - x^*\|_A$.]

Exercise 14 (*Convergence of the cg method*)

Assume that $A \in \mathbf{R}^{n \times n}$ is symmetric positive definite with $p < n$ different eigenvalues. Consider the linear system $Ax = b \in \mathbf{R}^n$ and show that in case of exact computations, for any initial iterate $x^0 \in \mathbf{R}^n$ the cg method terminates after at most p steps with the solution $A^{-1}b$.

[Hint: Apply the the Cayley-Hamilton theorem together with estimates for the iteration error.]

<u>Exercise 15</u> (*Preconditioned cg method*)

The numerical solution of two-point boundary value problems often gives rise to a symmetric matrix $A = (a_{ij})_{i,j=1}^n \in \mathbf{R}^{n\times n}$, where

$$a_{ij} \;=\; \begin{cases} 2 & \text{für} \quad i = j \\ -1 & \text{für} \quad |\, i - j \,| = 1 \\ 0 & \text{für} \quad |\, i - j \,| \geq 2 \end{cases} \quad , \quad 1 \leq i, j \leq n \ .$$

(i) Show that A is positive definite.

(ii) Let $D := \text{diag}(A)$ be the diagonal of A and denote by L the lower triangular part so that $A = L + D + L^T$.

Consider the linear system $Ax = b$, $b \in \mathbf{R}^n$ and show that the preconditioned cg method with preconditioner $C := EE^T$, where $E := \frac{1}{2}D + L$, converges after at most two steps.

[Hint: Observe that the convergence properties of the preconditioned cg method correspond to the original cg method applied to the transformed matrix $\tilde{A} := E^{-1}AE^{-T}$ and apply the result of Exercise 14.]

<u>Exercise 16</u> (*cg method for the discrete Laplacian*)

The discretization of the 2D Laplacian by finite differences with respect to a uniform grid results in a linear algebraic system with the symmetric coefficient matrix $A = (a_{\ell j})_{\ell,j=0}^{n-1} \in \mathbf{R}^{n\times n}, n = k^2$, $k \in \mathbf{N}$, where

$$a_{\ell j} \;=\; \begin{cases} 4 & , \quad \ell = j \\ -1 & , \quad (\ell \,\text{DIV}\, k) \;=\; (j \,\text{DIV}\, k) \wedge |\, \ell - j \,| = 1 \\ -1 & , \quad |\, \ell - j \,| = k \\ 0 & , \quad \text{otherwise} \end{cases} \quad , \quad 1 \leq \ell, j \leq n-1 \ .$$

Here, $\ell \,\text{DIV}\, k$ stands for the integer part of ℓ/k, whereas $\ell \,\text{MOD}\, k \in \{0, ..., k-1\}$ denotes the rest of the division.

(i) Represent A as a $k \times k$ block matrix and specify the individual blocks.

(ii) A vector $x = (x_0, ..., x_{n-1})^T \in \mathbf{R}^n$ can be stored in a $k \times k$ array $\hat{x}$ as follows:

$$x_\ell \;=\; \hat{x}[\ell \,\text{DIV}\, k \,,\, \ell \,\text{MOD}\, k] \quad , \quad 0 \leq \ell \leq n-1 \ .$$

Using this 'two-dimensional' ordering of the components of the vector, the action of the matrix A can be easily explained. Describe by means of a two-dimensional drawing how the components of $\hat{x}$ change under the mapping represented by the matrix A.

[Hint: $\hat{x}$ is a $k \times k$ array of certain values. In case of multiplication by A, a new value is assigned to each grid point which can be written as a weighted sum of other values. The weights in this sum correspond to the entries of A.]

(iii) Assume that for $1 \leq \mu, \nu \leq k$ the vectors $e_{\mu,\nu} \in \mathbf{R}^n$ are stored in the $k \times k$ arrays $\hat{e}_{\mu,\nu}$ of complex numbers. In this form, they can be represented according to

$$\hat{e}_{\mu,\nu}[i,j] \ := \ \sin\left(\pi\mu \frac{i+1}{k+1}\right) \sin\left(\pi\nu \frac{j+1}{k+1}\right) \quad , \quad 0 \leq i,j \leq k-1 \ .$$

Show that $e_{\mu,\nu}$ is an eigenvector of A corresponding to the eigenvalue

$$\lambda_{\mu,\nu} \ = \ 4\left(\sin^2\left(\frac{\pi}{2}\frac{\mu}{k+1}\right) + \sin^2\left(\frac{\pi}{2}\frac{\nu}{k+1}\right)\right) \ .$$

(iv) Compute $\lambda_{max}(A) := \max\{\mid \lambda \mid ; \lambda \in \sigma(A)\}$, $\lambda_{min}(A) := \min\{\mid \lambda \mid ; \lambda \in \sigma(A)\}$ and the spectral condition number

$$\kappa(A) \ := \ \frac{\lambda_{max}(A)}{\lambda_{min}(A)} \ .$$

(v) Show that A is positive definite.

(vi) Derive an estimate for the convergence rate of the cg method in the A energy norm. How does the convergence behave in terms of k?

Delivery of the homework at latest on October 2, 2009. The homework may be submitted either electronically (rohop@math.uh.edu) or as a hardcopy in class.