Department of Mathematics
University of Houston
Numerical Analysis I
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Numerical Mathematics I
(6th Homework Assignment)

Exercise 21 (*Regula falsi of higher order*)

Assume that $f \in C^1([a,c)), 0 < a < c$, has a simple zero $x^* \in [a,c]$ and that $f(a)f(c) < 0$. Let b such that $a < b < c$ and consider the quadratic polynomial $p \in P_2(\mathbb{R})$ interpolating f in a, b, c, i.e. $p(z) = f(z), z \in \{a, b, c\}$.

(i) Show that p has exactly one zero in $[a, c]$.

(ii) Denote by

$$\text{REAL zero}(a, b, c, f)$$

the function which computes the zero of p for given a, b, c. Using this function, construct an iterative algorithm which provides an approximation of x^* up to machine accuracy eps.

Exercise 22 (*Newton's method for the computation of the m-th root*)

(i) For $m \in \mathbb{N}$, the positive m-th root $x^* > 0$ of $a > 0$ is the solution of the equation

$$f(x) \;=\; 0 \quad \text{where } f(x) \;:=\; x^m \;-\; a \;.$$

Assume $m \geq 2$ and determine the Newton iteration

$$x^{(k+1)} \;=\; \Phi(x^{(k)})$$

for the computation of x^*.

(ii) Let f be a scalar real-valued function with zero x^* and denote by $x^{(k)}, k \in \mathbb{N}$, the iterates obtained by Newton's method. Show that there holds

$$x^* \;-\; x^{(k+1)} \;=\; -\,\frac{1}{2}\,\frac{f''(x^{(k)} + \theta(x^+ - x^{(k)}))}{f'(x^{(k)})}\,(x^* - x^{(k)})^2 \;,$$

where $0 \leq \theta \leq 1$.

(iii) For the particular function f from part (i) compute the asymptotic error constant

$$C \;:=\; \frac{1}{2}\,|\frac{f''(x^*)}{f'(x^*)}| \;.$$

What is its meaning with regard to the convergence of Newton's method close to the zero x^*?

<u>Exercise 23</u> (*Perturbation lemma*)

Given $B \in \mathbb{R}^{n\times n}$ and a submultiplicative matrix norm $\|\cdot\|$, assume $\|B\| < 1$. Show that the matrix $I - B$ is regular with

$$\|(I-B)^{-1}\| \;\le\; \frac{1}{1-\|B\|} \;.$$

<u>Exercise 24</u> (*Affine invariant convergence result for Newton's method*)

Let $F : D \subset \mathbb{R}^n \to \mathbb{R}^n, D$ convex, be continuously differentiable. Assume that $x^* \in D$ is a zero of F with regular Jaobi matrix $F'(x^*)$. Moreover, assume that for some $\omega^* > 0$ the following affine invariant Lipschitz condition holds true

$$\|F'(x^*)^{-1}(F'(y) - F'(x))\| \;\le\; \omega^*\|y-x\| \quad , \quad x, y \in D \;,$$

and that

$$B_\rho(x^*) \;:=\; \{x \in \mathbb{R}^n \mid \|x - x^*\| \le \rho\} \;\subset\; D \quad \text{for } \rho \;:=\; \|x^{(0)} - x^*\| \;<\; \frac{2}{3\omega^*}$$

for some initial value $x^{(0)}$.
If $\{x^{(k)}\}_{k\in\mathbb{N}_0}$ is the sequence of Newton iterates, show that $x^{(k)} \in B_\rho(x^*), k \in \mathbb{N}_0$, and that $x^{(k)} \to x^*$ as $k \to \infty$.
Moreover, show that x^* is the unique zero of F in D.

[Hint: Use induction on k and apply the perturbation lemma from Exercise 23 to verify that $F'(x^{(k)}), k \in \mathbb{N}_0$, is regular and $\|F'(x^{(k)})^{-1}F'(x^*)\| \le 3$.]

Delivery of the homework at latest on October 30, 2009. The homework may be submitted either electronically (rohop@math.uh.edu) or as a hardcopy in class.