



Department of Mathematics  
University of Houston  
Numerical Analysis I  
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## Numerical Mathematics I (7. Homework Assignment)

**Exercise 25** ( $L^2$ -estimate for piecewise linear interpolation)

The interpolation error estimates presented in class provide upper bounds for the pointwise error in polynomial interpolation. Often, it is of interest to have estimates of the error in specific norms as, for instance, the  $L^2$ -norm

$$\|f\|_{L^2([a,b])} := \left( \int_a^b |f(x)|^2 dx \right)^{1/2} .$$

Let  $\Delta_h := \{x_i := ih\}_{i=0}^n$ ,  $h := b/n$ ,  $n \in \mathbf{N}$  be a uniform partition of the interval  $[0, b]$  ( $b > 0$ ). For  $f \in C^2([0, b])$  we want to estimate the  $L^2$ -norm of the interpolation error  $f - s$ , where  $s$  represents the polygon which interpolates  $f$  at the nodes  $x_i$ ,  $0 \leq i \leq n$ .

(i) Show that for  $x \in [0, h]$  there holds

$$|f(x) - s(x)| \leq \sqrt{3}x\sqrt{h} \left( \int_0^h |f''(t)|^2 dt \right)^{1/2} .$$

[Hint: Use the integral representation of the remainder in the Taylor expansion, the Cauchy Schwarz inequality, and the fact that for a continuously differentiable function  $\Phi$  there holds  $\Phi(x) - \Phi(0) = \int_0^x \Phi'(t)dt$ .]

(ii) Show that

$$\int_0^h |f(x) - s(x)|^2 dx \leq h^4 \int_0^h |f''(x)|^2 dx .$$

(iii) Prove the  $L^2$  error estimate

$$\|f - s\|_{L^2([0,b])} \leq h^2 \|f''\|_{L^2([0,b])} .$$

**Exercise 26** (*Uniform convergence of interpolating polynomials*)

Let  $f(x) := \cos(x)$ . For each  $n \in \mathbb{N}$ , assume that  $x_i^{(n)} \in [0, \pi/2]$ ,  $0 \leq i \leq n$ , are pairwise different nodal points and denote by  $p_n(f) \in P_n([0, \pi/2])$  the associated interpolating polynomial.

Prove that the sequence  $\{p_n(f)\}_{\mathbb{N}}$  converges to  $f$  uniformly on  $[0, \pi/2]$ .

**Exercise 27** (*Non-uniform convergence of interpolating polynomials*)

Let  $f(x) = 1/(1+x^2)$ . Show that for any  $M > 0$  there exists  $n \in \mathbb{N}$  such that for the interpolating polynomial  $p_n(f) \in P_n$  with respect to  $(x_i, f(x_i))$ ,  $0 \leq i \leq n$ , where  $x_i = -1 + ih$ ,  $h = 1/n$ , there holds

$$\max_{-1 \leq x \leq +1} |f(x) - p_n(f)(x)| \geq M .$$

[Hinweis: Using the series expansion of  $f$  around some  $a \in [-1, +1]$  show that for  $k \rightarrow \infty$  the limes superior of  $f^{(k)}(a)/k!$  is positive. Then, use the representation of the remainder in polynomial interpolation to estimate the interpolation error from below at  $x = 1$ .]

**Exercise 28** (*Bernstein polynomials*)

(i) The Bernstein polynomials  $B_i^n \in P_n([0, 1])$ ,  $0 \leq i \leq n$ , are given by

$$B_i^n(\lambda) := \binom{n}{i} (1 - \lambda)^{n-i} \lambda^i .$$

Show that the Bernstein polynomials are nonnegative on  $[0, 1]$  and represent a partition of unity, i.e.,

$$B_i^n(\lambda) \geq 0 \quad , \quad \sum_{i=0}^n B_i^n(\lambda) = 1 \quad , \quad \lambda \in [0, 1] .$$

Moreover, show that the Bernstein polynomials satisfy the recursion

$$B_i^n(\lambda) = \lambda B_{i-1}^{n-1}(\lambda) + (1 - \lambda) B_i^{n-1}(\lambda) \quad , \quad \lambda \in [0, 1] \quad , \quad 1 \leq i \leq n$$

and provide a basis of the linear space  $P_n([0, 1])$ .

(ii) The linear space  $P_3([0, 1])$  of polynomials of degree  $\leq 3$  auf  $[0, 1]$  is spanned by the monomial basis  $\{x^i\}_{i=0}^3$ , and the Bernstein basis  $\{B_i^3\}_{i=0}^3$ . Compute the transformation matrix for a change of basis from the Bernstein basis to the monomial basis.

**Delivery of the homework at latest on November 13, 2009. The homework may be submitted either electronically (rohop@math.uh.edu) or as a hardcopy in class.**