Department of Mathematics
University of Houston
Numerical Analysis I
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# Numerical Mathematics I
# (9. Homework Assignment)

**Exercise 30** (*Perturbation lemma*)

Given $B \in \mathbb{R}^{n \times n}$ and a submultiplicative matrix norm $\| \cdot \|$, assume $\|B\| < 1$. Show that the matrix $I - B$ is regular with

$$\|(I - B)^{-1}\| \ \leq \ \frac{1}{1 - \|B\|} \ .$$

**Exercise 31** (*Affine invariant convergence result for Newton's method*)

Let $F : D \subset \mathbb{R}^n \to \mathbb{R}^n, D$ convex, be continuously differentiable. Assume that $x^* \in D$ is a zero of $F$ with regular Jaobi matrix $F'(x^*)$. Moreover, assume that for some $\omega^* > 0$ the following affine invariant Lipschitz condition holds true

$$\|F'(x^*)^{-1}(F'(y) - F'(x))\| \ \leq \ \omega^* \|y - x\| \quad , \quad x, y \in D \ ,$$

and that

$$B_\rho(x^*) \ := \ \{x \in \mathbb{R}^n \mid \|x - x^*\| \leq \rho\} \ \subset \ D \quad \text{for } \rho \ := \ \|x^{(0)} - x^*\| \ < \ \frac{2}{3\omega^*}$$

for some initial value $x^{(0)}$.
If $\{x^{(k)}\}_{k \in \mathbb{N}_0}$ is the sequence of Newton iterates, show that $x^{(k)} \in B_\rho(x^*), k \in \mathbb{N}_0$, and that $x^{(k)} \to x^*$ as $k \to \infty$.
Moreover, show that $x^*$ is the unique zero of $F$ in $D$.

[Hint: Use induction on $k$ and apply the perturbation lemma from Exercise 30 to verify that $F'(x^{(k)}), k \in \mathbb{N}_0$, is regular and $\|F'(x^{(k)})^{-1} F'(x^*)\| \leq 3$.]

**Exercise 32** (*Frobenius norm*)

The Frobenius norm $\|A\|_F$ of a matrix $A = (a_{ij}) \in \mathbb{R}^{m \times n}$ is given by

$$\|A\|_F \ := \ (\sum_{i=1}^{m} \sum_{j=1}^{n} a_{ij}^2)^{1/2} \ .$$

Show that

$$\|A\|_F^2 \;=\; \mathrm{tr}(A^TA) \;=\; \sum_{i=1}^{n} \lambda_i(A^TA) \;,$$

where $\mathrm{tr}(A^TA)$ is the trace of $A^TA \in \mathbb{R}^{n\times n}$ and $\lambda_i(A^TA), 1 \le i \le n$, are the eigenvalues of $A^TA$.

<u>**Exercise 33**</u> (*Good Broyden*)

The 'Good Broyden' is a Quasi-Newton method for the iterative solution of the nonlinear system $F(x) = 0$, where $F : D \subset \mathbb{R}^n \to \mathbb{R}^n$. Starting from an initial iterate $x^{(0)} \in D$ and an initial approximation $B_0$ of the Jacobian $F'(x^{(0)})$, a sequence $\{x^{(k)}\}_{k\in\mathbb{N}}$ of iterates is computed according to

$$x^{(k+1)} \;=\; x^{(k)} \;-\; B_k^{-1}F(x^{(k)}) \;.$$

Here, $B_{k+1}$ is obtained from $B_k$ such that

$$(+) \qquad B_{k+1}(x^{(k+1)} - x^{(k)}) \;=\; F(x^{(k+1)}) - F(x^{(k)}) \quad \text{(secant condition)} \;,$$

$$(++) \quad \|B_{k+1} - B_k\|_F \;=\; \min\{\|\tilde{B} - B_k\|_F \mid \tilde{B} \text{ satisfies the secant condition}\} \;,$$

where $\|\cdot\|_F$ stands for the Frobenius norm.
Show that the conditions (+) and (++) uniquely characterize the rank 1 update

$$B_{k+1} \;=\; B_k \;+\; \frac{(y^{(k)} - B_kp^{(k)})(p^{(k)})^T}{(p^{(k)})^Tp^{(k)}} \;,$$

$$y^{(k)} \;:=\; F(x^{(k+1)}) - F(x^{(k)}) \quad , \quad p^{(k)} \;:=\; x^{(k+1)} - x^{(k)} \;.$$

[Hint: Use the result of the previous exercise to interpret $B_{k+1}$ as the orthogonal projection onto an affine subspace of $\mathbb{R}^{n\times n}$.]

**Delivery of the homework at latest on November 22, 2005. The homework may be submitted either electronically (rohop@math.uh.edu) or as a hardcopy in class.**