



Department of Mathematics  
University of Houston  
Numerical Analysis I  
Dr. Ronald H.W. Hoppe



## Numerical Analysis I (2nd Practical Homework Assignment)

### Practical Exercise 2 (*cg method*)

The deflection  $u = u(x)$ ,  $x = (x_1, x_2)^T \in \Omega := (0, 1)^2$  of a clamped membrane due to an exterior force of force density  $f = f(x)$ ,  $x \in \Omega$  can be described by the boundary value problem

$$\begin{aligned} -\Delta u(x) &= -\left(\frac{\partial}{\partial x_1^2} u(x) + \frac{\partial}{\partial x_2^2} u(x)\right) = f(x) \quad , \quad x \in \Omega \\ u(x) &= 0 \quad , \quad x \in \Gamma := \partial\Omega . \end{aligned}$$

The approximation of the 2D Laplacian  $-\Delta := \frac{\partial}{\partial x_1^2} + \frac{\partial}{\partial x_2^2}$  by finite differences

$$\begin{aligned} \frac{\partial}{\partial x_1^2} u(x) &\sim \frac{u(x_1 - h, x_2) - 2u(x_1, x_2) + u(x_1 + h, x_2)}{h^2} , \\ \frac{\partial}{\partial x_2^2} u(x) &\sim \frac{u(x_1, x_2 - h) - 2u(x_1, x_2) + u(x_1, x_2 + h)}{h^2} \end{aligned}$$

with respect to a uniform grid of mesh size  $h > 0$  results in a linear algebraic system with the coefficient matrix  $A = (a_{ij})_{i,j=0}^{n-1} \in \mathbb{R}^{n \times n}$ ,  $n = k^2$ ,  $k \in \mathbb{N}$ , which is given by

$$a_{ij} = \begin{cases} 4 & ; \quad i = j \\ -1 & ; \quad (iDIVk) = (jDIVk) \text{ and } |i - j| = 1 \\ -1 & ; \quad |i - j| = k \\ 0 & ; \quad \text{otherwise} \end{cases} .$$

Here,  $iDIVk$  stands for the integer part of  $i/k$ .

Write a code which realizes the preconditioned cg method applied to the linear system  $Ax = b$  with the symmetric positive definite matrix  $A$  as given above.

As a termination criterion for the iteration use

$$\frac{\|r_i\|_2^2}{\|r_0\|_2^2} \leq \varepsilon ,$$

where  $r_i$  is the residual with respect to the  $i$ -th iteration and  $\|\cdot\|_2$  refers to the Euclidean norm. During the iteration, compute

$$\rho_i := \left( \frac{\|r_i\|_2}{\|r_1\|_2} \right)^{1/(i-1)} , \quad i \geq 2$$

as an estimate of the asymptotic convergence rate.

As the right-hand side  $b = (b_0, \dots, b_{n-1})^T$  choose  $b_i = 1$ ,  $0 \leq i \leq n-1$  and use the zero vector as the initial iterate. Further, choose  $\varepsilon = 10^{-7}$  as the tolerance in the termination criterion.

For  $k \in \{10, 20, 30, \dots, 100\}$ , compare the estimated asymptotic convergence rates and the total number of iterations in case of the

- cg method without preconditioning,
- cg method with the symmetric Gauss Seidel iteration as a preconditioner (SGS-pcg).

**Delivery at latest on October 16, 2009. Submit the following table containing the total number of iterations and computed asymptotic convergence rates**

k	cg		SGS-pcg	
	No. It.	$\rho$	No. It.	$\rho$
10				
20				
30				
·				
100				