

## Chapter 2 Sobolev spaces

In this chapter, we give a brief overview on basic results of the theory of Sobolev spaces and their associated trace and dual spaces.

### 2.1 Preliminaries

Let  $\Omega$  be a bounded domain in Euclidean space  $\mathbb{R}^d$ . We denote by  $\overline{\Omega}$  its closure and refer to  $\Gamma = \partial\Omega := \overline{\Omega} \setminus \Omega$  as its boundary. Moreover, we denote by  $\Omega_e := \mathbb{R}^d \setminus \overline{\Omega}$  the associated exterior domain.

We consider functions  $u : \Omega \rightarrow \mathbb{R}$  and denote by

$$D^\alpha u := \frac{\partial^{|\alpha|} u}{\partial x_1^{\alpha_1} \partial x_2^{\alpha_2} \dots \partial x_d^{\alpha_d}}$$

its partial derivatives of order  $|\alpha|$ . Here,  $\alpha = (\alpha_1, \dots, \alpha_d)^T \in \mathbb{N}_0^d$  is a multiindex of modulus  $|\alpha| = \sum_{i=1}^d \alpha_i$ .

We define by  $C^m(\Omega)$ ,  $m \in \mathbb{N}_0$ , the linear space of continuous functions on  $\Omega$  whose partial derivatives  $D^\alpha u$ ,  $|\alpha| \leq m$ , exist and are continuous.  $C^m(\Omega)$  is a Banach space with respect to the norm

$$\|u\|_{C^m(\Omega)} := \max_{0 \leq |\alpha| \leq m} \sup_{x \in \Omega} |D^\alpha u(x)|.$$

We refer to  $C^{m,\alpha}(\Omega)$ ,  $m \in \mathbb{N}_0$ ,  $0 < \alpha < 1$ , as the linear space of functions in  $C^m(\Omega)$  whose  $m$ -th order partial derivatives are Hölder continuous, i.e., for all  $\beta \in \mathbb{N}_0^d$  with  $|\beta| = m$  there exist constants  $\Gamma_\beta > 0$  such that for all  $x, y \in \Omega$

$$|D^\beta u(x) - D^\beta u(y)| \leq \Gamma_\beta |x - y|^\alpha.$$

We note that  $C^{m,\alpha}(\Omega)$  is a Banach space with respect to the norm

$$\|u\|_{C^{m,\alpha}(\Omega)} := \|u\|_{C^m(\Omega)} + \max_{|\beta|=m} \sup_{x,y \in \overline{\Omega}} \frac{|D^\beta u(x) - D^\beta u(y)|}{|x - y|^\alpha}.$$

Moreover,  $C_0^m(\Omega)$  and  $C_0^{m,\alpha}(\Omega)$  are the subspaces of functions with compact support in  $\Omega$ .

Finally,  $C^\infty(\Omega)$  stands for the set of functions with continuous partial derivatives of any order and  $C_0^\infty(\Omega)$  denotes the set of  $C^\infty(\Omega)$  functions with compact support in  $\Omega$ .

In the sequel, we will mainly deal with **Lipschitz domains** which are defined as follows.

**Definition 2.1 Lipschitz domain**

A bounded domain  $\Omega \subset \mathbb{R}^d$  with boundary  $\Gamma$  is said to be a **Lipschitz domain**, if there exist constants  $\alpha > 0, \beta > 0$ , and a finite number of **local coordinate systems**  $(x_1^r, x_2^r, \dots, x_d^r), 1 \leq r \leq R$ , and **local Lipschitz continuous mappings**

$$a_r : \{\hat{x}^r = (x_2^r, \dots, x_d^r) \in \mathbb{R}^{d-1} \mid |x_i^r| \leq \alpha, 2 \leq i \leq d\} \rightarrow \mathbb{R}$$

such that

$$\begin{aligned} \Gamma &= \bigcup_{r=1}^R \{(x_1^r, \hat{x}^r) \mid x_1^r = a_r(\hat{x}^r), |\hat{x}^r| < \alpha\}, \\ \{(x_1^r, \hat{x}^r) \mid a_r(\hat{x}^r) < x_1^r < a_r(\hat{x}^r) + \beta, |\hat{x}^r| < \alpha\} &\subset \Omega, 1 \leq r \leq R, \\ \{(x_1^r, \hat{x}^r) \mid a_r(\hat{x}^r) - \beta < x_1^r < a_r(\hat{x}^r), |\hat{x}^r| < \alpha\} &\subset \Omega_e, 1 \leq r \leq R, . \end{aligned}$$

In particular, the geometrical interpretation of the conditions is that both  $\Omega$  and  $\Omega_e$  are locally situated on exactly one side of the boundary  $\Gamma$ .

**Definition 2.2  $C^m$ -domain and  $C^{m,\alpha}$ -domain**

A Lipschitz domain  $\Omega \subset \mathbb{R}^d$  is a  **$C^m$ -domain** ( **$C^{m,\alpha}$ -domain**), if the functions  $a_r, 1 \leq r \leq R$ , in Definition 2.1 are  $C^m$ -functions ( $C^{m,\alpha}$ -functions).

**2.2 Sobolev spaces**

We refer to  $L^p(\Omega), p \in [1, \infty)$ , as the linear space of  $p$ -th order integrable functions on  $\Omega$  and to  $L^\infty(\Omega)$  as the linear space of essentially bounded functions which are Banach spaces with respect to the norms

$$\|v\|_{p,\Omega} := \left( \int_{\Omega} |v(x)|^p dx \right)^{1/p},$$

$$\|v\|_{\infty,\Omega} := \text{ess sup}_{x \in \Omega} |v(x)|.$$

Note that for  $p = 2$ , the space  $L^2(\Omega)$  is a Hilbert space with respect to the inner product

$$(v, w)_{0,\Omega} := \int_{\Omega} vw dx.$$

Sobolev spaces are based on the concept of weak (distributional) derivatives:

### Definition 2.3 Weak derivatives

Let  $u \in L^1(\Omega)$  and  $\alpha \in \mathbb{N}_0^d$ . The function  $u$  is said to have a **weak derivative**  $D_w^\alpha u$ , if there exists a function  $v \in L^1(\Omega)$  such that

$$\int_{\Omega} u D^\alpha \varphi \, dx = (-1)^{|\alpha|} \int_{\Omega} v \varphi \, dx \quad , \quad \varphi \in C_0^\infty(\Omega) .$$

We then set  $D_w^\alpha u := v$ .

The notion 'weak derivative' suggests that it is a generalization of the classical concept of differentiability and that there are functions which are weakly differentiable, but not differentiable in the classical sense. We give an example.

### Example 2.1 Example of a weakly differentiable function

Let  $d = 1$  and  $\Omega := (-1, +1)$ . The function  $u(x) := |x|, x \in \Omega$ , is not differentiable in the classical sense. However, it admits a weak derivative  $D_w^1 u$  given by

$$D_w^1 u = \begin{cases} -1 & , \quad x < 0 \\ +1 & , \quad x > 0 \end{cases} .$$

Indeed, for  $\varphi \in C_0^\infty(\Omega)$  we obtain by partial integration

$$\begin{aligned} \int_{-1}^{+1} u(x) D^1 \varphi(x) \, dx &= \int_{-1}^0 u(x) D^1 \varphi(x) \, dx + \int_0^{+1} u(x) D^1 \varphi(x) \, dx = \\ &= - \int_{-1}^0 D_w^1 u(x) \varphi(x) \, dx + (u\varphi)|_{-1}^0 - \int_0^{+1} D_w^1 u(x) \varphi(x) \, dx + (u\varphi)|_0^{+1} = \\ &= \int_{-1}^{+1} D_w^1 u(x) \varphi(x) \, dx - [u(0)] \varphi(0) , \end{aligned}$$

where  $[u(0)] := u(0+) - u(0-)$  is the jump of  $u$  in  $x = 0$ . But  $u$  is continuous and hence,  $[u(0)] = 0$  which allows to conclude.

### Definition 2.4 The Sobolev spaces $W^{m,p}(\Omega), p \in [1, \infty]$

The linear space  $W^{m,p}(\Omega)$  given by

$$(2.1) \quad W^{m,p}(\Omega) := \{ u \in L^p(\Omega) \mid D_w^\alpha u \in L^p(\Omega) , |\alpha| \leq m \}$$

is called a **Sobolev space**. It is a Banach space with respect to the norm

$$(2.2) \quad \|v\|_{m,p,\Omega} := \left( \sum_{|\alpha| \leq m} \|D_w^\alpha v\|_{p,\Omega}^p \right)^{1/p}, \quad p \in [1, \infty),$$

$$(2.3) \quad \|v\|_{m,\infty,\Omega} := \max_{|\alpha| \leq m} \|D_w^\alpha v\|_{\infty,\Omega}.$$

Note that  $W^{m,2}(\Omega)$  is a Hilbert space with respect to the inner product

$$(2.4) \quad (u, v)_{m,2,\Omega} := \sum_{|\alpha| \leq m} \int_{\Omega} D_w^\alpha u D_w^\alpha v \, dx.$$

The associated norm will be denoted by  $\|\cdot\|_{m,2,\Omega}$ . We will simply write  $(\cdot, \cdot)_{m,\Omega}$  and  $\|\cdot\|_{m,\Omega}$ , if the context clearly indicates that the  $W^{m,2}(\Omega)$ -inner product and the  $W^{m,2}(\Omega)$ -norm are meant.

A natural question is whether  $C^m(\Omega)$  is dense in  $W^{m,p}(\Omega)$ . To this end, we consider the linear space

$$C^{m,*}(\Omega) := \{ u \in C^m(\Omega) \mid \|u\|_{m,p,\Omega} < \infty \}.$$

It is a normed linear space with respect to  $\|\cdot\|_{m,p,\Omega}$ , but not complete, i.e., a pre-Banach space. We denote its completion with respect to the  $\|\cdot\|_{m,p,\Omega}$ -norm by  $H^{m,p}(\Omega)$ , i.e.,

$$(2.5) \quad H^{m,p}(\Omega) := \overline{C^{m,*}(\Omega)}^{\|\cdot\|_{m,p,\Omega}}.$$

We note that for  $p = \infty$  we have  $H^{m,\infty}(\Omega) = C^m(\Omega)$  and hence,  $W^{m,\infty}(\Omega) \neq H^{m,\infty}(\Omega)$ . However, a famous result by Meyers/Serrin (cf., e.g., [1]) states that for a Lipschitz domain and  $p \in [1, \infty)$  the spaces  $W^{m,p}(\Omega)$  and  $H^{m,p}(\Omega)$  are equivalent.

### Theorem 2.1 Characterization of Sobolev spaces

Let  $\Omega \subset \mathbb{R}^d$  be a Lipschitz domain and  $m \in \mathbb{N}_0$ . Then, for  $p \in [1, \infty)$  there holds

$$(2.6) \quad W^{m,p}(\Omega) = H^{m,p}(\Omega).$$

#### Remark 2.1 Remark on Meyers/Serrin

The result (2.6) actually holds true for any open subset  $\Omega \subset \mathbb{R}^d$ . However,  $C^m(\overline{\Omega})$  is dense in  $W^{m,p}(\Omega)$ , if  $\Omega$  satisfies the so-called **segment property** (cf., e.g., [1]). In particular, Lipschitz domains have this property.

Theorem 2.1 asserts that for Lipschitz domains and  $p \in [1, \infty)$  the space  $C^m(\Omega) \cap W^{m,p}(\Omega)$  is dense in  $W^{m,p}(\Omega)$ . A natural question to ask is whether or not functions in  $W^{m,p}(\Omega)$ ,  $m \geq 1$ , belong to the Banach

space  $L^q(\Omega)$ , or even are continuous. As we shall see, the latter only holds true, if  $m$  is sufficiently large (**Sobolev imbedding theorem**). However, let us first give an example which shows that in general we can not expect such a result:

**Example 2.2 Example of a weakly differentiable, but not essentially bounded function**

Let  $d \geq 2$ ,  $\Omega := \{x \in \mathbb{R}^d \mid |x| < \frac{1}{2}\}$  and consider the function

$$u(x) := \ln(|\ln(|x|)|).$$

The function  $u$  has square-integrable first-order weak derivatives

$$D^\alpha u(x) = \frac{x^\alpha}{|x|^2 \ln(|x|)} \quad , \quad |\alpha| = 1,$$

since in view of

$$|D^\alpha u(x)|^d \leq \rho(|x|) := \frac{1}{|x|^d |\ln(|x|)|^d} \quad , \quad |\alpha| = 1,$$

it possesses a square-integrable majorant.

On the other hand,  $u$  obviously is not essentially bounded.

**Theorem 2.2 Sobolev imbedding theorems**

Let  $\Omega \subset \mathbb{R}^d$  be a Lipschitz domain,  $m \in \mathbb{N}_0$  and  $p \in [1, \infty]$ . Then, the following mappings represent continuous imbeddings

$$(2.7) \quad W^{m,p}(\Omega) \rightarrow L^{p^*}(\Omega) \quad , \quad \frac{1}{p^*} = \frac{1}{p} - \frac{m}{d} \quad , \quad \text{if } m < \frac{d}{p} \quad ,$$

$$(2.8) \quad W^{m,p}(\Omega) \rightarrow L^q(\Omega) \quad , \quad q \in [1, \infty) \quad , \quad \text{if } m = \frac{d}{p} \quad ,$$

$$(2.9) \quad W^{m,p}(\Omega) \rightarrow C^{0, m - \frac{d}{p}}(\bar{\Omega}) \quad , \quad \text{if } \frac{d}{p} < m < \frac{d}{p} + 1 \quad ,$$

$$(2.10) \quad W^{m,p}(\Omega) \rightarrow C^{0,\alpha}(\bar{\Omega}) \quad , \quad 0 < \alpha < 1 \quad , \quad \text{if } m = \frac{d}{p} + 1 \quad ,$$

$$(2.11) \quad W^{m,p}(\Omega) \rightarrow C^{0,1}(\bar{\Omega}) \quad , \quad \text{if } m > \frac{d}{p} + 1 \quad .$$

**Proof.** For a proof, we refer to [1]. □

In the following chapters, we will frequently take advantage of **compact imbeddings** of Sobolev spaces.

**Theorem 2.3 Kondrasov imbedding theorems**

Let  $\Omega \subset \mathbb{R}^d$  be a Lipschitz domain,  $m \in \mathbb{N}_0$  and  $p \in [1, \infty]$ . Then, the following mappings are compact imbeddings

$$(2.12) \quad W^{m,p}(\Omega) \rightarrow L^q(\Omega) \quad , \quad 1 \leq q \leq p^* \quad , \quad \frac{1}{p^*} = \frac{1}{p} - \frac{m}{d} \quad , \quad \text{if } m < \frac{d}{p} \quad ,$$

$$(2.13) \quad W^{m,p}(\Omega) \rightarrow L^q(\Omega) \quad , \quad q \in [1, \infty) \quad , \quad \text{if } m = \frac{d}{p} \quad ,$$

$$(2.14) \quad W^{m,p}(\Omega) \rightarrow C^0(\bar{\Omega}) \quad , \quad \text{if } m > \frac{d}{p} \quad .$$

**Proof.** The interested reader is referred to [1]. □

We finally deal with **restrictions** and **extensions** of Sobolev functions:

If  $u \in W^{m,p}(\mathbb{R}^d)$ ,  $p \in [1, \infty]$ , then for any domain  $\Omega \subset \mathbb{R}^d$  the restriction  $Ru$  of  $u$  to  $\Omega$

$$Ru(x) := u(x) \quad \text{f.a.a. } x \in \Omega$$

belongs to  $W^{m,p}(\Omega)$  and  $R : W^{m,p}(\mathbb{R}^d) \rightarrow W^{m,p}(\Omega)$  is a bounded linear operator.

Conversely, it is in general not possible to continuously extend a function  $u \in W^{m,p}(\Omega)$  to a function in  $W^{m,p}(\mathbb{R}^d)$  as the following example shows:

**Example 2.3 Counterexample (Extension of Sobolev functions)**

Let  $d = 2$  ,  $\Omega := \{x = (x_1, x_2) \in \mathbb{R}^2 \mid 0 < x_1 < 1, |x_2| < x_1^r\}$ ,  $r > 1$ , and consider the function

$$u(x) := x_1^{-\frac{\varepsilon}{p}} \quad , \quad 0 < \varepsilon < r \quad .$$

We note that  $\Omega$  has a cusp in the origin and hence, is no Lipschitz domain.

For  $\varepsilon < r + 1 - p$ , we have that  $u \in W^{1,p}(\Omega)$ , since

$$\sum_{|\alpha|=1} \int_{\Omega} |D^\alpha u|^p dx = C_{\varepsilon,p} \int_0^1 x_1^{-\varepsilon-p+r} dx_1 \quad .$$

On the other hand,  $u$  is not in  $L^\infty(\Omega)$ . Choosing  $\varepsilon$  such that  $p > 2$  is possible, we see that a **Lipschitz domain** is **necessary** for the Sobolev imbedding theorem to hold true. Since the Sobolev imbedding

theorem is valid on  $\mathbb{R}^d$ , we can not extend  $u$  to a function in  $W^{1,2}(\mathbb{R}^2)$ .

However, in case of a Lipschitz domain we have the validity of the following Sobolev extension theorem.

**Theorem 2.3 Sobolev extension theorem**

Let  $\Omega \subset \mathbb{R}^d$  be a Lipschitz domain,  $m \in \mathbb{N}_0$ , and  $p \in [1, \infty]$ . Then, there exists a bounded linear extension operator  $E : W^{m,p}(\Omega) \rightarrow W^{m,p}(\mathbb{R}^d)$ , i.e.,  $Eu = u$  for all  $u \in W^{m,p}(\Omega)$  and there exists a constant  $C \geq 0$  such that for all  $u \in W^{m,p}(\Omega)$

$$\|Eu\|_{m,p,\Omega} \leq C \|u\|_{m,p,\mathbb{R}^d} .$$

**Proof.** We refer to [3]. □

We recall that the **algebraic and topological dual**  $V^*$  of a Hilbert space  $V$  is the linear space of all bounded linear functionals on  $V$  which is itself a Hilbert space with respect to the norm

$$\|\ell\|_{V^*} := \sup_{v \neq 0} \frac{|\ell(v)|}{\|v\|_V} .$$

We may thus define the dual spaces of the Sobolev spaces  $W^{m,p}(\Omega)$ ,  $p \in [1, \infty]$ .

**Definition 2.5 Sobolev spaces with negative index**

Let  $\Omega \subset \mathbb{R}^d$  be a bounded domain, let  $m$  be a negative integer and suppose  $p \in [1, \infty]$ . Then, the Sobolev space  $W^{m,p}(\Omega)$  is defined as the dual space  $(W^{-m,q}(\Omega))^*$ , where  $q$  is conjugate to  $p$ , i.e.,  $\frac{1}{q} + \frac{1}{p} = 1$ .

**Remark 2.2 The Dirac  $\delta$ -function**

The Sobolev spaces  $W^{m,p}(\Omega)$ ,  $m < 0$ , are proper subspaces of  $L^p(\Omega)$ . For instance, for  $m < -d + \frac{d}{p}$ , if  $p < \infty$ , and  $m \leq -d$ , if  $p = \infty$ , they contain the Dirac  $\delta$ -function considered as a linear functional

$$\begin{aligned} \delta : W^{-m,p}(\Omega) &\rightarrow \mathbb{R} \\ u &\longmapsto \delta_x(u) , \end{aligned}$$

where  $x$  is some given point in  $\Omega$ .

## 2.3 Sobolev spaces with broken index

For  $\Omega = \mathbb{R}^d$ , we define **Sobolev spaces with broken index**  $H^s(\mathbb{R}^d)$ ,  $s \in \mathbb{R}_+$  by using the **Fourier transform**  $\hat{v}$  of a function  $v \in C_0^\infty(\mathbb{R}^d)$

$$\hat{v}(\xi) := \left(\frac{1}{2\pi}\right)^{d/2} \int_{\mathbb{R}^d} \exp(-i\xi \cdot x) v(x) dx .$$

The associated **Sobolev norm** is defined according to

$$(2.15) \quad \|v\|_{s, \mathbb{R}^d} := \|(1 + |\cdot|^2)^{s/2} \hat{v}(\cdot)\|_{0, \mathbb{R}^d} ,$$

and we set

$$(2.16) \quad H^s(\mathbb{R}^d) := \overline{C_0^\infty(\mathbb{R}^d)}^{\|\cdot\|_{s, \mathbb{R}^d}} .$$

If  $\Omega \subset \mathbb{R}^d$  is a Lipschitz domain, the space  $H^s(\Omega)$  can be either defined implicitly by means of

$$(2.17) \quad \|v\|_{s, \Omega} := \inf_{z=Ev \in H^s(\mathbb{R}^d)} \|z\|_{s, \mathbb{R}^d} ,$$

where  $Ev$  is the **extension** of  $v$  to  $H^s(\mathbb{R}^d)$ , or - for  $s = m + \lambda$ ,  $m \in \mathbb{N}_0$ ,  $0 \leq \lambda < 1$ , - explicitly by

$$(2.18) \quad \|v\|_{s, \Omega} := \left( \left(\frac{1}{\text{diam}(\Omega)}\right)^{2s} \|v\|_{0, \Omega}^2 + |v|_{s, \Omega}^2 \right)^{1/2} ,$$

where  $|\cdot|_{\lambda, \Omega}$  stands for the **seminorm**

$$(2.19) \quad |v|_{s, \Omega}^2 := \sum_{1 \leq |\alpha| \leq m} \|D^\alpha v\|_{0, \Omega}^2 + \sum_{|\alpha|=m} \int_{\Omega} \int_{\Omega} \frac{|D^\alpha v(x) - D^\alpha v(y)|^2}{|x - y|^{d+2\lambda}} dx dy .$$

If  $\Sigma \subseteq \Gamma = \partial\Omega$ , we define the space  $H^s(\Sigma)$  as follows

$$(2.20) \quad H^s(\Sigma) := \{v \in L^2(\Sigma) \mid \|v\|_{s, \Sigma} < \infty\} ,$$

equipped with the norm

$$(2.21) \quad \|v\|_{s, \Sigma} := \left( \left(\frac{1}{\text{diam}(\Sigma)}\right)^{2s} \|v\|_{0, \Sigma}^2 + |v|_{s, \Sigma}^2 \right)^{1/2} ,$$

where

$$(2.22) \quad |v|_{s, \Sigma}^2 := \sum_{1 \leq |\alpha| \leq m} \|D^\alpha v\|_{0, \Sigma}^2 + \sum_{|\alpha|=m} \int_{\Sigma} \int_{\Sigma} \frac{|D^\alpha v(x) - D^\alpha v(y)|^2}{|x - y|^{d-1+2\lambda}} d\sigma_x d\sigma_y .$$

In case  $\Sigma \subset \Gamma$  and  $v \in H^s(\Sigma)$ ,  $s < 1$ , we define  $\tilde{v}$  as the **extension by zero** of  $v$ , i.e.,

$$(2.23) \quad \tilde{v}(x) := \begin{cases} v(x) & , \quad x \in \Sigma \\ 0 & , \quad x \in \Gamma \setminus \Sigma \end{cases} .$$

We define  $H_{00}^s(\Sigma)$  according to

$$(2.24) \quad H_{00}^s(\Sigma) := \{v \in L^2(\Sigma) \mid \tilde{v} \in H^s(\Gamma)\} ,$$

equipped with the norm

$$(2.25) \quad \|v\|_{H_{00}^s(\Sigma)} := \left( \|v\|_{0,\Sigma}^2 + |v|_{H_{00}^s(\Sigma)}^2 \right)^{1/2} ,$$

where

$$(2.26) \quad |v|_{H_{00}^s(\Sigma)}^2 := |v|_{s,\Sigma}^2 + \int_{\Sigma} \frac{v^2(x)}{\text{dist}(x, \partial\Sigma)} d\sigma$$

and

$$(2.27) \quad |v|_{s,\Sigma}^2 := \int_{\Sigma} \int_{\Sigma} \frac{|v(x) - v(y)|^2}{|x - y|^{d-1+2s}} d\sigma_x d\sigma_y .$$

For  $s < 0$ , we define  $\tilde{H}^s(G)$ ,  $G \in \{\Sigma, \Omega\}$ , as the dual of  $H^{-s}(G)$ , equipped with the **negative norm**

$$(2.28) \quad |v|_{s,G} := \sup_{w \in H^{-s}(G), w \neq 0} \frac{\langle v, w \rangle_G}{\|w\|_{-s,G}} ,$$

where  $\langle \cdot, \cdot \rangle_G$  stands for the dual pairing.

Moreover, if  $-1 < s < 0$ , we define  $H^s(G)$ ,  $G \in \{\Sigma, \Omega\}$ ,  $\Sigma \subset \Gamma$ , as the dual of  $H_{00}^{-s}(G)$ . For  $s = -1$ , we further define  $H^{-1}(G)$  as the dual of  $H_0^1(G)$ , whereas for  $\Sigma = \Gamma$  and  $-1 \leq s < 0$  we define  $H^s(\Gamma)$  as the dual of  $H^{-s}(\Gamma)$ .

For details we refer to [2].

## 2.4 Trace spaces

For a bounded domain  $\Omega \subset \mathbb{R}^d$  with boundary  $\Gamma$  and a function  $u \in C(\overline{\Omega})$ , it makes sense to define the restriction of  $u$  to the boundary  $\Gamma$  simply by considering the pointwise restriction. In view of the Sobolev imbedding theorem, however, the **pointwise restriction** does not make sense for functions  $u \in W^{m,2}(\Omega)$  unless  $m$  is sufficiently large. Therefore, we have to find appropriate means to specify the **trace**  $u|_{\Gamma}$  of a function  $u \in W^{m,2}(\Omega)$ . The following example suggests how to proceed.

**Example 2.4 Trace of smooth functions**

Let  $\Omega \subset \mathbb{R}^2$  be the unit disk which in polar coordinates reads as follows

$$\Omega := \{ (r, \theta) \mid 0 \leq r < 1, 0 \leq \theta < 2\pi \}.$$

For  $u \in C^1(\overline{\Omega})$ , the restriction to  $\partial\Omega$  can be expressed according to

$$\begin{aligned} u(1, \theta)^2 &= \int_0^1 \frac{\partial}{\partial r} (r^2 u(r, \theta)^2) dr = \int_0^1 2(r^2 u u_r + r u^2)(r, \theta) dr = \\ &= \int_0^1 2 \left( r^2 u \nabla u \cdot \frac{(x_1, x_2)}{r} + r u^2 \right)(r, \theta) dr \leq \\ &\leq \int_0^1 2 (r^2 |u| |\nabla u| + r u^2)(r, \theta) dr \leq \\ &\leq \int_0^1 2 (|u| |\nabla u| + u^2)(r, \theta) r dr. \end{aligned}$$

Integrating the previous inequality over  $\theta$  and applying the Cauchy-Schwarz inequality as well as Young's inequality results in

$$\begin{aligned} \|u\|_{0, \partial\Omega}^2 &= \int_{\partial\Omega} u^2 d\theta \leq 2 \int_{\Omega} (|u| |\nabla u| + u^2) dx_1 dx_2 \leq \\ &\leq 2 \left( \int_{\Omega} u^2 dx \right)^{1/2} \left( \int_{\Omega} |\nabla u|^2 dx \right)^{1/2} + 2 \int_{\Omega} u^2 dx \leq \\ &\leq \sqrt[4]{8} \|u\|_{0, \Omega}^{1/2} \|u\|_{1, \Omega}^{1/2}. \end{aligned}$$

The example can be easily generalized to give the following result.

**Lemma 2.1 Trace of Sobolev space functions. Part I**

Let  $\Omega \subset \mathbb{R}^2$  be the unit disk and  $u \in W^{1,2}(\Omega)$ . Then, the trace  $u|_{\partial\Omega}$  can be interpreted as a function in  $L^2(\partial\Omega)$  satisfying

$$(2.29) \quad \|u\|_{0, \partial\Omega}^2 \leq \sqrt[4]{8} \|u\|_{0, \Omega}^{1/2} \|u\|_{1, \Omega}^{1/2}.$$

**Proof.** Since  $C^1(\overline{\Omega})$  is dense in  $W^{1,2}(\Omega)$ , there exists a sequence  $(u_k)_{\mathbb{N}} \subset C^1(\overline{\Omega})$  such that  $\|u - u_k\|_{1, \Omega} \leq 1/k, k \in \mathbb{N}$ . Applying the

inequality derived in Example 2.4, we obtain

$$\begin{aligned} \|u_i - u_k\|_{0,\partial\Omega} &\leq \sqrt[4]{8} \|u_i - u_k\|_{0,\Omega}^{1/2} \|u_i - u_k\|_{1,\Omega}^{1/2} \leq \\ &\leq \sqrt[4]{8} \|u_i - u_k\|_{1,\Omega} \leq \sqrt[4]{8} \left( \frac{1}{i} + \frac{1}{k} \right). \end{aligned}$$

Consequently,  $(u_k)_{\mathbb{N}}$  is a Cauchy sequence in  $L^2(\partial\Omega)$  and hence, there exists  $v \in L^2(\partial\Omega)$  such that  $u_k \rightarrow v$  ( $k \rightarrow \infty$ ) in  $L^2(\partial\Omega)$ .

We define the **trace** of  $u$  on  $\partial\Omega$  according to

$$u_{\partial\Omega} := v.$$

We have to show that  $u_{\partial\Omega}$  is well defined, i.e., it does not depend on the particular choice of a sequence of  $C^1(\bar{\Omega})$ -functions. To this end, let  $(v_k)_{\mathbb{N}}, v_k \in C^1(\bar{\Omega}), k \in \mathbb{N}$  be another sequence satisfying  $\|u - v_k\|_{1,\Omega} \rightarrow 0$  ( $k \rightarrow \infty$ ). We find

$$\begin{aligned} \|v - v_k\|_{0,\partial\Omega} &\leq \|v - u_k\|_{0,\partial\Omega} + \|u_k - v_k\|_{0,\partial\Omega} \leq \\ &\leq \|v - u_k\|_{0,\partial\Omega} + \sqrt[4]{8} \|u_k - v_k\|_{1,\Omega} \leq \\ &\leq \|v - u_k\|_{0,\partial\Omega} + \sqrt[4]{8} \left( \|u_k - u\|_{1,\Omega} + \|u - v_k\|_{1,\Omega} \right) \\ &\rightarrow 0 \quad (k \rightarrow \infty). \end{aligned}$$

Finally, the validity of (2.29) follows by a density argument:

$$\begin{aligned} \|u\|_{0,\partial\Omega} &= \|v\|_{0,\partial\Omega} = \lim_{k \rightarrow \infty} \|u_k\|_{0,\partial\Omega} \leq \\ &\leq \lim_{k \rightarrow \infty} \sqrt[4]{8} \|u_k\|_{0,\partial\Omega}^{1/2} \|u_k\|_{1,\Omega}^{1/2} = \sqrt[4]{8} \|u\|_{0,\partial\Omega}^{1/2} \|u\|_{1,\Omega}^{1/2}. \quad \square \end{aligned}$$

The previous result can be generalized to arbitrary Lipschitz domains.

### Theorem 2.4 Trace of Sobolev space functions. Part II

Let  $\Omega \subset \mathbb{R}^d$  be a Lipschitz domain and  $p \in [1, \infty]$ . Then, the trace mapping

$$\begin{aligned} W^{1,p}(\Omega) &\rightarrow L^p(\partial\Omega) \\ u &\mapsto u|_{\partial\Omega} \end{aligned}$$

is a bounded linear, injective mapping.

**Proof.** We refer to [2]. □

However, the trace mapping considered as a mapping from  $W^{1,p}(\Omega)$  in  $L^p(\partial\Omega)$  is not surjective, i.e., there exist functions  $v \in L^p(\partial\Omega)$  such that we can not find  $u \in W^{1,p}(\Omega)$  with  $u|_{\partial\Omega} = v$ .

In particular, for  $p = 2$  we have the following result.

**Theorem 2.5 Trace of Sobolev space functions. Part III**

Let  $\Omega \subset \mathbb{R}^d$  be a Lipschitz domain and  $1/2 < s \leq 1$ . Then there holds:

(i) There exists a **unique linear bounded trace mapping**

$$(2.30) \quad \begin{aligned} \gamma_0 : H^s(\Omega) &\rightarrow H^{s-1/2}(\Gamma) \\ v &\longmapsto \gamma_0(v) := v|_{\Gamma} \end{aligned}$$

such that

$$(2.31) \quad \begin{aligned} \|\gamma_0(v)\|_{s-1/2,\Gamma} &\leq C \|v\|_{s,\Omega} , \\ |\gamma_0(v)|_{s-1/2,\Gamma} &\leq C |v|_{s,\Omega} . \end{aligned}$$

(ii) The **trace operator**  $\gamma_0$  has a **bounded right inverse**

$$(2.32) \quad \gamma_0^- : H^{s-1/2}(\Gamma) \rightarrow H^s(\Omega) ,$$

i.e.,  $\gamma_0 \gamma_0^- w = w$ ,  $w \in H^{s-1/2}(\Gamma)$ , and

$$(2.33) \quad \begin{aligned} \|\gamma_0^-(w)\|_{s,\Omega} &\leq C \|w\|_{s-1/2,\Gamma} , \\ |\gamma_0^-(w)|_{s,\Omega} &\leq C |w|_{s-1/2,\Gamma} . \end{aligned}$$

(iii) In case  $\Sigma \subset \Gamma$ , the results in (i) and (ii) hold true with  $H^{s-1/2}(\Gamma)$  replaced by  $H_{00}^{s-1/2}(\Sigma)$  and  $H^s(\Omega)$  replaced by  $H_{\Gamma_D}^s(\Omega)$ , where  $\Gamma_D := \Gamma \setminus \Sigma$  and

$$(2.34) \quad H_{\Gamma_D}^s(\Omega) := \{v \in H^s(\Omega) \mid \gamma_0 v = 0 \text{ on } \Gamma_D\} .$$

**Proof.** We refer to [2]. □

**Definition 2.6 Dirichlet data**

For  $u \in H^1(\Omega)$ , the trace  $\gamma_0 u$  is called the **Dirichlet data**.

**Definition 2.7 Normal component trace mapping**

Given a Lipschitz domain  $\Omega \subset \mathbb{R}^d$  with boundary  $\Gamma$ , we denote by

$$\mathbf{n} = (n_1, \dots, n_d)^T$$

the **unit exterior normal vector** which is well defined at almost all  $x \in \Gamma$ . For a vector field  $\mathbf{q} \in C^\infty(\overline{\Omega})^d$ , we refer to

$$(2.35) \quad \eta_{\mathbf{n}}(\mathbf{q}) := \mathbf{n} \cdot \mathbf{q}|_{\Gamma} .$$

as the **normal component trace mapping**. It can be extended by continuity to a linear continuous mapping

$$(2.36) \quad \eta_{\mathbf{n}} : H^1(\Omega)^d \rightarrow H^{1/2}(\Gamma) .$$

**Definition 2.8 Normal derivative and Neumann data**

Within the notations of Definition 2.7, for  $u \in C^\infty(\overline{\Omega})$  the normal component trace of  $\nabla u$

$$(2.37) \quad \partial_{\mathbf{n}}u := \eta_{\mathbf{n}}(\nabla u) = \mathbf{n} \cdot \nabla u$$

is called the **normal derivative** of  $u$ .

The mapping  $\partial_{\mathbf{n}}$  can be extended to a linear continuous mapping

$$(2.38) \quad \partial_{\mathbf{n}} : H^2(\Omega) \rightarrow H^{1/2}(\Gamma) .$$

In particular, for  $u \in H^2(\Omega)$  the normal derivative  $\partial_{\mathbf{n}}u$  is called the **Neumann data**.

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