Chapter 5 A priori error estimates for nonconforming finite element approximations

5.1 Strang's first lemma

We consider the variational equation

$$(5.1)\qquad a(u,v) \;=\; \ell(v) \quad , \quad v \in V \subset H^1(\Omega) \;,$$

and assume that the conditions of the Lax-Milgram lemma are satisfied so that (5.1) admits a unique solution $u \in V$.
We assume that $\mathcal{H}$ is a null sequence of positive real numbers and $(V_h)_{h\in\mathcal{H}}$ an associated **family of conforming finite element spaces** $V_h \subset V, h \in \mathcal{H}$. We approximate the bilinear form $a(\cdot,\cdot) : V \times V \to \mathbb{R}$ and the functional $\ell(\cdot) : V \to \mathbb{R}$ in (5.1) by bounded bilinear forms and bounded linear functionals

$$(5.2)\qquad a_h(\cdot,\cdot) : V_h \times V_h \to \mathbb{R} \quad , \quad \ell_h(\cdot) : V_h \to \mathbb{R} \quad , \quad h \in \mathcal{H}$$

and consider the finite dimensional variational equations

$$(5.3)\qquad a_h(u_h,v_h) \;=\; \ell_h(v_h) \quad , \quad v_h \in V_h \;,\; h \in \mathcal{H} \;.$$

Definition 5.1 Uniform V_h-ellipticity

The sequence $(a_h(\cdot,\cdot))_{h\in\mathcal{H}}$ of approximate bilinear forms $a_h(\cdot,\cdot) : V_h \times V_h \to \mathbb{R}$ is called **uniformly V_h-elliptic**, if there exists a positive constant $\tilde{\alpha}$ such that uniformly in $h \in \mathcal{H}$

$$(5.4)\qquad a_h(u_h,u_h) \;\geq\; \tilde{\alpha}\; \|u_h\|_V^2 \quad , \quad u_h \in V_h \;.$$

Under the assumption of uniform V_h-ellipticity, the variational equations (5.3) admit unique solutions $u - h \in V_h$. The following result, known as **Strang's first lemma**, can be viewed as a generalization of Céa's lemma.

Theorem 5.1 Strang's first lemma

Assume that $(a_h(\cdot,\cdot))_{h\in\mathcal{H}}$ is a **uniformly V_h-elliptic** family of bilinear forms and denote by $u \in V$ and $u_h \in V_h, h \in \mathcal{H}$ the unique solutions of (5.1) and (5.3), respectively. then, there exists a constant $C \in \mathbb{R}_+$, independent of $h \in \mathcal{H}$ such that

$$(5.5)\quad \|u - u_h\|_V \;\leq$$
$$\leq\; C\, \Big(\inf_{v_h \in V_h} \Big(\|u - v_h\|_V \;+\; \sup_{w_h \in V_h} \frac{|a(v_h,w_h) - a_h(v_h,w_h)|}{\|w_h\|_V} \Big) \;+$$
$$+\; \sup_{w_h \in V_h} \frac{|\ell(w_h) - \ell_h(w_h)|}{\|w_h\|_V} \Big) \quad , \quad h \in \mathcal{H} \;.$$

Proof. Taking advantage of the uniform V_h-ellipticity, for $v_h \in V_h$ we obtain

$$\tilde{\alpha}\ \|u_h - v_h\|_V^2 \ \le\ a_h(u_h - v_h, u_h - v_h)\ \pm a(u - v_h, u_h - v_h)\ =$$
$$= \ a(u - v_h, u_h - v_h)\ +\ \Big(a(v_h, u_h - v_h)\ -\ a_h(v_h, u_h - v_h)\Big)\ +$$
$$+\Big(\ell_h(u_h - v_h)\ -\ \ell(u_h - v_h)\Big)\ .$$

Using the boundedness of the bilinear form, i.e., $|a(u,v)| \le M\|u\|_V\|v\|_V$, and dividing the previous inequality by $\|u_h - v_h\|_V$, it follows that

$$\begin{aligned}\tilde{\alpha}\|u_h - v_h\|_V^2 \ &\le\ M\|u_h - v_h\|_V + \frac{|a(v_h, u_h - v_h) - a_h(v_h, u_h - v_h)|}{\|u_h - v_h\|_V} + \\ &\quad + \frac{|\ell_h(u_h - v_h) - \ell(u_h - v_h)|}{\|u_h - v_h\|_V}\ \le \\ &\le\ M\ \|u_h - v_h\|_V + \sup_{w_h \in V_h} \frac{|a(v_h, w_h) - a_h(v_h, w_h)|}{\|w_h\|_V}\ + \\ &\quad +\ \sup_{w_h \in V_h} \frac{|\ell_h(w_h) - \ell(w_h)|}{\|w_h\|_V}\ .\end{aligned}$$

Using the triangle inequality

$$\|u - u_h\|_V\ \le\ \|u - v_h\|_V\ +\ \|u_h - v_h\|_V$$

and the previous inequality, we deduce (5.5). □

Strang's first lemma shows that the upper bound for the global discretization error consists of two parts: the **approximation error**

$$\inf_{v_h \in V_h} \|u - v_h\|_V \tag{5.6}$$

and the **consistency errors**

$$\inf_{v_h \in V_h} \sup_{w_h \in V_h} \frac{|a(v_h, w_h) - a_h(v_h, w_h)|}{\|w_h\|_V}\ , \tag{5.7}$$

$$\sup_{w_h \in V_h} \frac{|\ell(w_h) - \ell_h(w_h)|}{\|w_h\|_V}\ . \tag{5.8}$$

As an application of **Strang's first lemma**, in the subsequent section we will study the effect of **numerical integration** in the finite element approximation of second order elliptic boundary value problems.

5.2 Numerical integration

5.2.1 Construction of quadrature formulas

We consider the **model variational equation**

$$a(u,v) \;=\; \ell(v) \quad , \quad v \in V := H_0^1(\Omega) \;, \tag{5.9}$$

where $\Omega \subset \mathbb{R}^d$ is supposed to be a polygonal resp. polyhedral domain and the bilinear form $a(\cdot,\cdot) : V \times V \to \mathbb{R}$ and the functional $\ell(\cdot) : V \to \mathbb{R}$ are given according to

$$a(u,v) \;:=\; \int_\Omega \sum_{i,j=1}^d a_{ij} \, \frac{\partial u}{\partial x_i} \frac{\partial v}{\partial x_j} \; dx \;, \tag{5.10}$$

and

$$\ell(v) \;:=\; \int_\Omega f v \; dx \;. \tag{5.11}$$

Here, $f \in L^2(\Omega)$ and $a_{ij} \in C(\overline{\Omega}), 1 \leq i,j \leq d$, satisfying

$$\sum_{i,j=1}^d a_{ij}(x) \, \xi_i \xi_j \;\geq\; \alpha \sum_{i=1}^d \xi_i^2 \quad , \quad x \in \overline{\Omega} \;,\; \alpha > 0 \;. \tag{5.12}$$

We approximate (5.9) using a finite element space V_h generated by a triangulation $\mathcal{T}_h$ and finite elements $(K, P_K, \Sigma_K), K \in \mathcal{T}_h$, satisfying the assumptions **(A1),(A2),(A3)** of section 4.3 and $\hat{P}_{\hat{K}} \subset H^1(\hat{K})$ which implies **conformity**, i.e., $V_h \subset V$.
We approximate the bilinear form (5.10) and the functional (5.11) by evaluating the integrals by means of appropriate **quadrature formulas** with respect to the individual elements $K \in \mathcal{T}_h$ of the triangulation. Taking into account the **affine equivalence** of the finite elements with respect to a **reference element** $\hat{K}$

$$\begin{aligned} F_K \;:\; \hat{K} \;&\to\; K \\ \hat{x} \;&\to\; F_K(\hat{x}) = B_K \hat{x} + b_K \quad , \quad K \in \mathcal{T}_h \;, \end{aligned}$$

due to the **transformation of variables formula**

$$\int_K \varphi(x) \; dx \;=\; \det(B_K) \int_{\hat{K}} \hat{\varphi}(\hat{x}) \; d\hat{x} \quad , \quad \hat{\varphi} := \varphi \circ F_K, \tag{5.13}$$

a quadrature formula for an integral over the reference element $\hat{K}$ induces a suitable quadrature formula for the associated integral over the

actual element $K = F_K(\hat{K})$ of $\mathcal{T}_h$.
In particular, we consider the **reference quadrature formula**

$$\hat{Q}_{\hat{K}}(\hat{\varphi}) \ := \ \sum_{\ell=1}^{L} \hat{\omega}_{\ell,\hat{K}} \ \hat{\varphi}(\hat{b}_{\ell,\hat{K}}) \ , \tag{5.14}$$

with **weights** $\hat{\omega}_{\ell,\hat{K}}, 1 \leq \ell \leq L$, and **nodes** $\hat{b}_{\ell,\hat{K}} \in \hat{K}, 1 \leq \ell \leq L$.
We refer to

$$\hat{E}_{\hat{K}}(\hat{\varphi}) \ := \ \int_{\hat{K}} \hat{\varphi}(\hat{x}) \ d\hat{x} \ - \ \hat{Q}_{\hat{K}}(\hat{\varphi}) \tag{5.15}$$

as the associated **quadrature error functional**.

Lemma 5.1 Quadrature formulas in the FEM

Assume that $\hat{Q}_{\hat{K}}$ is a quadrature formula with respect to the reference element $\hat{K}$ as given by (5.14). Then, for the affine equivalent element $K := F_K(\hat{K})$ we obtain a quadrature formula according to

$$Q_K(\varphi) \ := \ \sum_{i=1}^{L} \omega_{i,K} \ \varphi(b_{i,K}) \ , \tag{5.16}$$

$$\omega_{\ell,K} \ := \ \det(B_K) \ \hat{\omega}_{\ell,\hat{K}} \quad , \quad b_{\ell,K} \ := \ F_K(\hat{b}_{\ell,\hat{K}}) \ , \ 1 \leq \ell \leq L \ .$$

Moreover, referring to

$$E_K(\varphi) \ := \ \int_K \varphi(x) \ dx \ - \ Q_K(\varphi) \tag{5.17}$$

as the related quadrature error functional, we have

$$E_K(\varphi) \ = \ \det(B_K) \ \hat{E}_{\hat{K}}(\hat{\varphi}) \ . \tag{5.18}$$

Proof. The proof is a direct consequence of (5.13). □

Based on the quadrature formula (5.16), we define the **approximate bilinear forms** $a_h(\cdot,\cdot) : V_h \times V_h \to \mathbb{R}$ by means of

$$a_h(u_h, v_h) \ := \ \sum_{K \in \mathcal{T}_h} \sum_{\ell=1}^{L} \omega_{\ell,K} \sum_{u,j=1}^{d} (a_{ij} \ \frac{\partial u_h}{\partial x_i} \ \frac{\partial v_h}{\partial x_j})(b_{\ell,K}) \ , \tag{5.19}$$

and the **approximate linear functionals** $\ell_h(\cdot) : V_h \to \mathbb{R}$ according to

$$\ell_h(v_h) \ := \ \sum_{K \in \mathcal{T}_h} \sum_{\ell=1}^{L} \omega_{\ell,K} \ (f v_h)(b_{\ell,K}) \ . \tag{5.20}$$

Then, the **finite element approximation** of the model problem (5.9) amounts to the computation of $u_h \in V_h, h \in \mathcal{H}$, as the solution of the approximate variational equation

$$a_h(u_h, v_h) \;=\; \ell_h(v_h) \;,\; v_h \in V_h. \tag{5.21}$$

We note that according to the Lax-Milgram lemma, (5.21) admits a unique solution, if the bilinear form $(a_h(\cdot,\cdot))_{h\in\mathcal{H}}$ is bounded and V_h-elliptic.

We first derive **sufficient conditions** for the **uniform V_h-ellipticity** of the family $(a_h(\cdot,\cdot))_{h\in\mathcal{H}}$ of approximate bilinear forms.

Theorem 5.2 Sufficient conditions for uniform V_h-ellipticity

Let $\hat{Q}_{\hat{K}}$ be a quadrature formula with respect to the reference element $\hat{K}$ with positive weights $\hat{\omega}_{\ell,\hat{K}}, 1 \le \ell \le L$, and the property that there exists $q \in \mathbb{N}$ such that

- $\hat{P}_{\hat{K}} \subset P_q(\hat{K})$,
- **(i)** the quadrature formula is exact for polynomials $\hat{p} \in P_{2q-2}(\hat{K})$, or
  (ii) the union of nodes $\bigcup_{\ell=1}^{L}\{\hat{b}_{\ell,\hat{K}}\}$ contains a $P_{q-1}(\hat{K})$-unisolvent subset.

Then, there exists a positive constant $\tilde{\alpha}$, independent of $h \in \mathcal{H}$, such that

$$a_h(v_h, v_h) \;\ge\; \tilde{\alpha}\; |v_h|_{1,\Omega}^2 \quad , \quad v_h \in V_h \;,\; h \in \mathcal{H} \;. \tag{5.22}$$

Proof. Observing $v_h|_K = p_K \in P_K$ and using the ellipticity condition (5.12), we find

$$\begin{aligned} &\sum_{\ell=1}^{L} \omega_{\ell,K} \sum_{i,j=1}^{d} (a_{ij}\, \frac{\partial v_h}{\partial x_i}\, \frac{\partial v_h}{\partial x_j})(b_{\ell,K}) \;= \\ =\; &\sum_{\ell=1}^{L} \omega_{\ell,K} \sum_{i,j=1}^{d} (a_{ij}\, \frac{\partial p_K}{\partial x_i}\, \frac{\partial p_K}{\partial x_j})(b_{\ell,K}) \;\ge \\ \ge\; &\alpha \sum_{\ell=1}^{L} \omega_{\ell,K} \underbrace{\sum_{i=1}^{d} (\frac{\partial p_K}{\partial x_i}(b_{\ell,K}))^2}_{=\;\|Dp_K(b_{\ell,K})\|^2} \;. \end{aligned} \tag{5.23}$$

Now, observing

$$D\hat{p}_{\hat{K}}(\hat{b}_{\ell,\hat{K}})\xi \;=\; Dp(b_{\ell,K})(B_K\xi) \quad , \quad 1 \le \ell \le L \;,$$

we have

$$\|D\hat{p}_{\hat{K}}(\hat{b}_{\ell,\hat{K}})\| \ \le \|B_K\| \ \|Dp(b_{\ell,K})\| \quad , \quad 1 \le \ell \le L \ ,$$

and hence, using Theorem 4.3, we get

$$\sum_{\ell=1}^{L} \omega_{\ell,K} \underbrace{\sum_{i=1}^{d} (\frac{\partial p_K}{\partial x_i}(b_{\ell,K}))^2}_{= \ \|Dp_K(b_{\ell,K})\|^2} \ge \|B_K\|^{-2} \sum_{\ell=1}^{L} \omega_{\ell,K} \sum_{i=1}^{d} (\frac{\partial \hat{p}_{\hat{K}}}{\partial \hat{x}_i}(\hat{b}_{\ell,\hat{K}}))^2 \tag{5.24}$$

$$= \det(B_K) \ \|B_K\|^{-2} \sum_{\ell=1}^{L} \hat{\omega}_{\ell,\hat{K}} \sum_{i=1}^{d} (\frac{\partial \hat{p}_{\hat{K}}}{\partial \hat{x}_i}(\hat{b}_{\ell,\hat{K}}))^2 \ .$$

(i) Let us first assume that the quadrature formula $\hat{Q}_{\hat{K}}$ is exact for polynomials $\hat{p} \in P_{2q-2}(\hat{K})$. Since

$$\sum_{i=1}^{d} (\frac{\partial \hat{p}_{\hat{K}}}{\partial \hat{x}_i})^2 \in P_{2q-2}(\hat{K}) \ ,$$

we then have

$$|\hat{p}_{\hat{K}}|_{1,\hat{K}}^2 \ = \ \int_{\hat{K}} \sum_{i=1}^{d} (\frac{\partial \hat{p}_{\hat{K}}}{\partial \hat{x}_i})^2 \ d\hat{x} \ = \ \sum_{i=1}^{d} (\frac{\partial \hat{p}_{\hat{K}}}{\partial \hat{x}_i}(\hat{b}_{\ell,\hat{K}}))^2 \ . \tag{5.25}$$

Inserting (5.25) into (5.24) and using again Theorem 4.3, it follows that

$$\begin{aligned} &\sum_{\ell=1}^{L} \omega_{\ell,K} \sum_{i=1}^{d} (\frac{\partial p_K}{\partial x_i}(b_{\ell,K}))^2 \ \ge \\ &\ge \ \det(B_K) \ \|B_K\|^{-2} \sum_{\ell=1}^{L} \hat{\omega}_{\ell,\hat{K}} \sum_{i=1}^{d} (\frac{\partial \hat{p}_{\hat{K}}}{\partial \hat{x}_i}(\hat{b}_{\ell,\hat{K}}))^2 \ = \\ &= \ \det(B_K) \ \|B_K\|^{-2} \ |\hat{p}_{\hat{K}}|_{1,\hat{K}}^2 \ \ge \\ &\ge \ \Big(\|B_K\| \ \|B_K^{-1}\|\Big)^{-2} \ |p_K|_{1,K}^2 \ . \end{aligned} \tag{5.26}$$

Due to the shape regularity of $\mathcal{T}_h, h \in \mathcal{H}$,

$$\|B_K\| \ \|B_K^{-1}\| \ \le \ \frac{\hat{h}_{\hat{k}}}{\hat{\rho}_{\hat{K}}} \ \frac{h_k}{\rho_K} \ \le \ C \ .$$

Combining (5.23),(5.24) and (5.26), we deduce the existence of a positive constant $\tilde{\alpha}$, independent of $h \in \mathcal{H}$, such that

$$\sum_{\ell=1}^{L} \omega_{\ell,K} \sum_{i,j=1}^{d} (a_{ij} \ \frac{\partial v_h}{\partial x_i} \ \frac{\partial v_h}{\partial x_j})(b_{\ell,K}) \ \ge \ \tilde{\alpha} \ |v_h|_{1,K}^2 \quad , \quad v_h \in V_h \ . \tag{5.27}$$

Forming the sum over all $K \in \mathcal{T}_h$ in (5.27), gives the assertion.

(ii) What remains to be shown is that the assertion also holds true, if we assume that the union $\bigcup_{\ell=1}^{L} \{\hat{b}_{\ell,\hat{K}}\}$ contains a $P_{q-1}(\hat{K})$-unisolvent subset. We claim that in this case (5.25) has to be replaced by

$$\sum_{i=1}^{d} \Big(\frac{\partial \hat{p}_{\hat{K}}}{\partial \hat{x}_i}(\hat{b}_{\ell,\hat{K}})\Big)^2 \;\ge\; \hat{C}\; |\hat{p}_{\hat{K}}|^2_{1,\hat{K}} \;, \tag{5.28}$$

where $\hat{C}$ is a positive constant. The proof then proceeds in the same way as before. In order to verify (5.28), it suffices to show that

$$\Big(\sum_{\ell=1}^{L} \hat{\omega}_{\ell,\hat{K}} \sum_{i=1}^{d} \Big(\frac{\partial \hat{p}_{\hat{K}}}{\partial \hat{x}_i}(\hat{b}_{\ell,\hat{K}})\Big)^2\Big)^{1/2}$$

provides a norm on the quotient space $\hat{P}_{\hat{K}}/P_0(\hat{K})$, since so does $|\cdot|_{1,\hat{K}}$, and we may conclude taking advantage of the equivalence of norms on finite dimensional spaces.

For that purpose, we assume

$$\sum_{\ell=1}^{L} \hat{\omega}_{\ell,\hat{K}} \sum_{i=1}^{d} \Big(\frac{\partial \hat{p}_{\hat{K}}}{\partial \hat{x}_i}(\hat{b}_{\ell,\hat{K}})\Big)^2 \;=\; 0 \;.$$

Then, the positivity of the weights $\hat{\omega}_{\ell,\hat{K}}, 1 \le \ell \le L$, yields

$$\frac{\partial \hat{p}_{\hat{K}}}{\partial \hat{x}_i}(\hat{b}_{\ell,\hat{K}}) \;=\; 0 \quad , \quad 1 \le i \le d \;,\; 1 \le \ell \le L \;.$$

But for each $i \in \{1, ..., d\}$

$$\frac{\partial \hat{p}_{\hat{K}}}{\partial \hat{x}_i} \in P_{q-1}(\hat{K}) \;,$$

and hence,

$$\frac{\partial \hat{p}_{\hat{K}}}{\partial \hat{x}_i} \;\equiv\; 0 \;,$$

since it vanishes on a $P_{q-1}(\hat{K})$-unisolvent subset. □

With Theorem 5.2 at hand, we may apply **Strang's first lemma**. If the solution $u \in V$ of (5.9) satisfies $u \in V \cap H^{k+1}(\Omega), k \in \mathbb{N}$,, for the **approximation error** we get

$$\inf_{v_h \in V_h} \|u - v_h\|_{1,\Omega} \;\le\; \|u - \Pi_h u\|_{1,\Omega} \;\le\; C\; h^k\; |u|_{k+1,\Omega} \;. \tag{5.29}$$

Therefore, we have to provide **sufficient conditions** ensuring that the **consistency error** does not deteriorate this order of convergence, i.e., we are looking for **consistency error estimates** of the form

$$\sup_{w_h \in V_h} \frac{|a(v_h,w_h) - a_h(v_h,w_h)|}{\|w_h\|_V} \;\le\; C(a_{ij},u)\; h^k \;, \tag{5.30}$$

$$\sup_{w_h \in V_h} \frac{|\ell(w_h) - \ell_h(w_h)|}{\|w_h\|_V} \;\le\; C(\ell)\; h^k \;, \tag{5.31}$$

where $C(a_{ij},u)$ and $C(\ell)$ are positive constants, independent of $h \in \mathcal{H}$.

The following auxiliary result will be used to establish such consistency error estimates.

Lemma 5.2 Generalized Leibniz formula

Assume that $\Omega \subset \mathbb{R}^d$ is a Lipschitz domain and $u \in H^m(\Omega)$ and $v \in W^{m,\infty}(\Omega)$ for some $m \in \mathbb{N}_0$. Then, there exists a positive constant $C(m,d)$ such that

$$|uv|_{m,\Omega} \;\le\; C(m,d) \sum_{j=0}^{m} |u|_{m-j,\Omega}\; |v|_{j,\infty,\Omega} \;. \tag{5.32}$$

Proof. The proof is left as an exercise. □

Theorem 5.3 Consistency error estimate I

Let $\hat{Q}_{\hat{K}}$ be a quadrature formula with respect to the reference element $\hat{K}$ with positive weights $\hat{\omega}_{\ell,\hat{K}}, 1 \le \ell \le L$, and the property that there exists $k \in \mathbb{N}$ such that

$$\hat{P}_{\hat{K}} = P_k(\hat{K}) \;, \tag{5.33}$$

$$\hat{E}_{\hat{K}}(\hat{\varphi}) = 0 \;,\; \hat{\varphi} \in P_{2k-2}(\hat{K}) \;. \tag{5.34}$$

Then, there exists a positive constant C, independent of $h \in \mathcal{H}$, such that for all $a \in W^{k,\infty}(\Omega)$ and all $p,q \in P_k(K)$

$$|E_K(a\frac{\partial q}{\partial x_i}\frac{\partial p}{\partial x_j})| \;\le\; C\; h_K^k\; \|a\|_{k,\infty,K}\; |p|_{1,K}\; \|q\|_{k,K} \;. \tag{5.35}$$

Proof. Since $\frac{\partial q}{\partial x_i}, \frac{\partial p}{\partial x_j} \in P_{k-1}(K)$, it suffices to prove (5.35) for

$$E_K(avw) \quad , \quad a \in W^{k,\infty}(K) \;,\; v,w \in P_{k-1}(K) \;.$$

In view of (5.18) in Lemma 5.1, we have

$$E_K(avw) \;=\; \det(B_K)\; \hat{E}_{\hat{K}}(\hat{a}\hat{v}\hat{w})$$

where

$$\hat{a} \in W^{k,\infty}(\hat{K}) \quad , \quad \hat{v},\hat{w} \in P_{k-1}(\hat{K}) \;.$$

(i) Since $\hat{a}\hat{v} \in W^{k,\infty}(\hat{K})$, we first provide an estimate for

$$\hat{E}_{\hat{K}}(\hat{\varphi}\hat{w}) \quad , \quad \hat{\varphi} \in W^{k,\infty}(\hat{K}) \ , \ \hat{w} \in P_{k-1}(\hat{K}) \ .$$

By direct estimation and using the equivalence of norms on $P_{k-1}(\hat{K})$, we obtain

$$|\hat{E}_{\hat{K}}(\hat{\varphi}\hat{w})| \ = \ | \int_{\hat{K}} \hat{\varphi}\hat{w} \ d\hat{x} \ - \ \sum_{\ell=1}^{L} \hat{\omega}_{\ell,\hat{K}} \ (\hat{\varphi}\hat{w})(\hat{b}_{\ell,\hat{K}})| \ \le$$

$$\le \ \hat{C} \ |\hat{\varphi}\hat{w}|_{0,\infty,\hat{K}} \ \le \ \hat{C} \ |\hat{\varphi}|_{0,\infty,\hat{K}} \ |\hat{w}|_{0,\infty,\hat{K}} \ \le \ \hat{C} \ \|\hat{\varphi}\|_{k,\infty,\hat{K}} \ |\hat{w}|_{0,\hat{K}} \ ,$$

which proves the continuity of the functional $\hat{E}_{\hat{K}}(\hat{w}) : W^{k,\infty}(\hat{K}) \to \mathbb{R}$ given by $(\hat{E}_{\hat{K}}(\hat{w}))(\hat{\varphi}) := \hat{E}_{\hat{K}}(\hat{\varphi}\hat{w})$ with

$$\|\hat{E}_{\hat{K}}(\hat{w})\| \ \le \ \hat{C} \ \|\hat{w}\|_{0,\hat{K}} \ . \tag{5.36}$$

Since $P_{k-1}(\hat{K}) \subset \mathrm{Ker}(\hat{E}_{\hat{K}}(\hat{w}))$, using (5.36) in the **Bramble-Hilbert lemma**, we get

$$|\hat{E}_{\hat{K}}(\hat{\varphi}\hat{w})| \ \le \ \hat{C} \ |\hat{\varphi}|_{k,\infty,\hat{K}} \ \|\hat{w}\|_{0,\hat{K}} \ . \tag{5.37}$$

(ii) We now consider the case $\hat{\varphi} = \hat{a}\hat{v}$ where $\hat{a} \in W^{k,\infty}(\hat{K})$ and $\hat{v} \in P_{k-1}(\hat{K})$. In view of the **generalized Leibniz formula** (cf. Lemma 5.2) and taking $|\hat{v}|_{k,\infty,\hat{K}} = 0$ into account, we obtain

$$|\hat{E}_{\hat{K}}(\hat{a}\hat{v}\hat{w})| \ \le \ \hat{C} \ \Bigl(\sum_{j=1}^{k-1} |\hat{a}|_{k-j,\infty,\hat{K}} \ |\hat{v}|_{j,\hat{K}} \Bigr) \ \|\hat{w}\|_{0,\hat{K}} \ . \tag{5.38}$$

Now, Theorem 4.3 tells us

$$|\hat{a}|_{k-j,\infty,\hat{K}} \ \le \ \hat{C} \ h_K^{k-j} \ |a|_{k-j,\infty,K} \quad , \quad 0 \le j \le k-1 \ , \tag{5.39}$$

$$|\hat{v}|_{j,\hat{K}} \ \le \ \hat{C} \ |\det(B_K)|^{-1/2} \ h_K^j \ |v|_{j,K} \quad , \quad 0 \le j \le k-1 \ , \tag{5.40}$$

$$\|\hat{w}\|_{0,\hat{K}} \ \le \ \hat{C} \ |\det(B_K)|^{-1/2} \ \|w\|_{0,K} \ . \tag{5.41}$$

Using (5.38)-(5.41), we finally obtain

$$|E_K(avw)| \ \le \ C \ h_K^k \ \Bigl(\sum_{j=0}^{k-1} |a|_{k-j,\infty,K} \ |v|_{j,K} \Bigr) \ \|w\|_{0,K} \ \le$$

$$\le \ C \ h_K^k \ \|a\|_{k,\infty,K} \ \|v\|_{k-1,K} \ \|w\|_{0,K} \ . \quad \square$$

Theorem 5.4 Consistency error estimate II

Let $\hat{Q}_{\hat{K}}$ be a quadrature formula with respect to the reference element $\hat{K}$ with positive weights $\hat{\omega}_{\ell,\hat{K}}, 1 \le \ell \le L$, and the property that there exists $k \in \mathbb{N}$ such that

$$\hat{P}_{\hat{K}} = P_k(\hat{K}) \ , \tag{5.42}$$

$$\hat{E}_{\hat{K}}(\hat{\varphi}) = 0 \ , \ \hat{\varphi} \in P_{2k-2}(\hat{K}) \ . \tag{5.43}$$

Assume that $g \in W^{k,p}(\Omega)$ and $p \in P_k(K)$ and suppose that $k \in \mathbb{N}$ satisfies

$$k \ > \ \frac{d}{p} \ . \tag{5.44}$$

Then, there exists a constant $C \in \mathbb{R}_+$, independent of $h \in \mathcal{H}$, such that for all $K \in \mathcal{T}_h$

$$|E_K(gp)| \ \le \ C \ (\text{meas}(K))^{(\frac{1}{2}-\frac{1}{p})} \ h_K^k \ \|g\|_{k,p,K} \ \|p\|_{1,K} \ . \tag{5.45}$$

Proof. In view of (5.18) in Lemma 5.1, we have

$$E_K(gp) \ = \ \det(B_K) \ \hat{E}_{\hat{K}}(\hat{g}\hat{p})$$

where

$$\hat{g} \in W^{k,p}(\hat{K}) \quad , \quad \hat{p} \in P_k(\hat{K}) \ .$$

Denoting by $\hat{Q} \, : \, L^2(\hat{K}) \to P_1(\hat{K})$ the **orthogonal projection** of $L^2(\hat{K})$ onto $P_k(\hat{K})$, we split $\hat{E}_{\hat{K}}(\hat{g}\hat{p})$ according to

$$\hat{E}_{\hat{K}}(\hat{g}\hat{p}) \ = \ \hat{E}_{\hat{K}}(\hat{g}\hat{Q}\hat{p}) \ + \ \hat{E}_{\hat{K}}(\hat{g}(\hat{p} - \hat{Q}\hat{p})) \tag{5.46}$$

and estimate both parts separately. (Note that a direct estimation (without such a splitting) would result in a non optimal estimate).

(i) Estimation of the first term in (5.46)

In view of (5.44) and the **Sobolev imbedding theorem**, the space $W^{k,p}(\hat{K})$ is continuously imbedded in $C^0(\hat{K})$. Hence, for $\hat{\psi} \in W^{k,p}(\hat{K})$

$$|\hat{E}_{\hat{K}}(\hat{\psi})| \ \le \ \hat{C} \ |\hat{\psi}|_{0,\infty,\hat{K}} \ \le \ \hat{C} \ \|\hat{\psi}\|_{k,p,\hat{K}} \ .$$

Since

$$\hat{E}_{\hat{K}}(\hat{\psi}) \ = \ 0 \quad , \quad \hat{\psi} \in P_{k-1}(\hat{K}) \ ,$$

the **Bramble-Hilbert lemma** infers

$$|\hat{E}_{\hat{K}}(\hat{\psi})| \ \le \ \hat{C} \ |\hat{\psi}|_{k,p,\hat{K}} \ . \tag{5.47}$$

Now, for $\hat{\psi} = \hat{g}\hat{Q}\hat{p}$, observing that

$$|\hat{Q}\hat{p}|_{\ell,\infty,\hat{K}} \ = \ 0 \quad , \quad \ell \ge 2 \ ,$$

the **generalized Leibniz formula** implies

$$|\hat{g}\hat{Q}\hat{p}|_{k,p,\hat{K}} \ \le \ \hat{C} \ \Big(|\hat{g}|_{k,p,\hat{K}} \ |\hat{Q}\hat{p}|_{0,\infty,\hat{K}} \ + \ |\hat{g}|_{k-1,p,\hat{K}} \ |\hat{Q}\hat{p}|_{1,\infty,\hat{K}} \Big) \ .$$

Taking advantage of the equivalence of norms on $P_1(\hat{K})$, we get

$$|\hat{g}\hat{Q}\hat{p}|_{k,p,\hat{K}} \ \le \ \hat{C} \ \Big(|\hat{g}|_{k,p,\hat{K}} \ |\hat{Q}\hat{p}|_{0,\hat{K}} \ + \ |\hat{g}|_{k-1,p,\hat{K}} \ |\hat{Q}\hat{p}|_{1,\hat{K}} \Big) \ . \tag{5.48}$$

Since $\hat{Q}$ is an orthogonal projection, we have

$$|\hat{Q}\hat{p}|_{0,\hat{K}} \ \le \ |\hat{p}|_{0,\hat{K}} \ . \tag{5.49}$$

Moreover, since $\hat{Q}\hat{p} = \hat{p} \ , \ \hat{p} \in P_0(\hat{K})$, the **Bramble-Hilbert lemma** gives

$$|\hat{p} - \hat{Q}\hat{p}|_{1,\hat{K}} \ \le \ \hat{C} \ |\hat{p}|_{1,\hat{K}} \ ,$$

whence

$$|\hat{Q}\hat{p}|_{1,\hat{K}} \ \le \ |\hat{p} - \hat{Q}\hat{p}|_{1,\hat{K}} + |\hat{p}|_{1,\hat{K}} \le \ (1 + \hat{C}) \ |\hat{p}|_{1,\hat{K}} \ . \tag{5.50}$$

Combining (5.47)-(5.50), we obtain

$$\hat{E}_{\hat{K}}(\hat{g}\hat{Q}\hat{p})| \le \ \hat{C} \ \Big(|\hat{g}|_{k,p,\hat{K}} |\hat{p}|_{0,\hat{K}} + |\hat{g}|_{k-1,p,\hat{K}} |\hat{p}|_{1,\hat{K}} \Big) \ . \tag{5.51}$$

(ii) Estimation of the second term in (5.46)

Since $\hat{p} - \hat{Q}\hat{p} = 0$ for $\hat{p} \in P_1(\hat{K})$, we may assume $k \ge 2$.
We further choose $r \in [1,\infty)$ large enough so that the mappings

$$W^{k,p}(\hat{K}) \ \to \ W^{k-1,r}(\hat{K}) \quad , \quad W^{k-1,r}(\hat{K}) \ \to \ C^0(\hat{K})$$

represent continuous imbeddings.

Case 1 $1 \le p < d$

We may choose $\frac{1}{r} = \frac{1}{p} - \frac{1}{d}$ so that by the **Sobolev imbedding theorem** the mapping $W^{1,p}(\hat{K}) \to L^r(\hat{K})$ and hence, also the mapping $W^{k,p}(\hat{K}) \to W^{k-1,r}(\hat{K})$ is a continuous imbedding.
Moreover, in view of (5.44), we have

$$k - 1 - \frac{d}{r} \ = \ k - \frac{d}{p} \ > \ 0 \ .$$

Then, the **Sobolev imbedding theorem** also guarantees that the mapping $W^{k-1,r}(\hat{K}) \to C^0(\hat{K})$ is a continuous imbedding.

Case 2 $d < p$

In this case, due to the **Sobolev imbedding theorem** the mapping $W^{1,p}(\hat{K}) \to L^r(\hat{K})$ is a continuous imbedding for all $r \in [1,\infty]$, if $d < p$, and for all $r \in [1,\infty)$, if $d = p$. It follows that the same holds

true for the mapping $W^{k,p}(\hat{K}) \to W^{k-1,r}(\hat{K})$.
Further, choosing r so large that

$$k - 1 - \frac{d}{r} \; > \; 0 \; ,$$

the **Sobolev imbedding theorem** implies that $W^{k-1,r}(\hat{K}) \to C^0(\hat{K})$ is a continuous imbedding.

Now, in either case we obtain

$$\begin{aligned} |\hat{E}_{\hat{K}}(\hat{g}(\hat{p} - \hat{Q}\hat{p}))| \; &\leq \; \hat{C} \; |\hat{g}(\hat{p} - \hat{Q}\hat{p})|_{0,\infty,\hat{K}} \; \leq \\ &\leq \; \hat{C} \; |\hat{g}|_{0,\infty,\hat{K}} \; |\hat{p} - \hat{Q}\hat{p}|_{0,\infty,\hat{K}} \; \leq \\ &\leq \; \hat{C} \; \|\hat{g}\|_{k-1,r,\hat{K}} \; |\hat{p} - \hat{Q}\hat{p}|_{0,\infty,\hat{K}} \; . \end{aligned}$$

Consequently, the linear functional $\hat{E}_{\hat{K}}(\hat{p}) : W^{k-1,r}(\hat{K}) \to \mathbb{R}$ given by $(\hat{E}_{\hat{K}}(\hat{p}))(\hat{g}) := \hat{E}_{\hat{K}}(\hat{g}(\hat{p} - \hat{Q}\hat{p}))$ is continuous with

$$\|\hat{E}_{\hat{K}}(\hat{p})\| \; \leq \; \hat{C} \; \|\hat{p} - \hat{Q}\hat{p}\|_{0,\infty,\hat{K}} \; .$$

Since

$$\hat{E}_{\hat{K}}(\hat{p}) \; = \; 0 \quad , \quad \hat{p} \in P_{k-2}(\hat{K}) \; ,$$

by the **Bramble-Hilbert lemma**

$$|\hat{E}_{\hat{K}}(\hat{g}(\hat{p} - \hat{Q}\hat{p}))| \; \leq \; \hat{C} \; |\hat{g}|_{k-1,r,\hat{K}} \; |\hat{p} - \hat{Q}\hat{p}|_{0,\hat{K}} \; . \tag{5.52}$$

Moreover, $\hat{Q}\hat{p} = \hat{p}, \hat{p} \in P_0(\hat{K})$, and hence,

$$|\hat{p} - \hat{Q}\hat{p}|_{o,\hat{K}} \; \leq \; \hat{C} \; |\hat{p}|_{1,\hat{K}} \; . \tag{5.53}$$

In order to estimate $\hat{g}|_{k-1,r,\hat{K}}$ in (5.52), we note that in view of the **Sobolev imbedding theorem** the mapping

$$W^{1,p}(\hat{k}) \; \to \; L^r(\hat{k})$$

is a continuous imbedding, whence

$$|\hat{f}|_{0,r,\hat{K}} \; \leq \; \hat{C} \; \Big(|\hat{f}|_{0,p,\hat{K}} \; + \; |\hat{f}|_{1,p,\hat{K}} \Big) \; ,$$

and hence,

$$|\hat{g}|_{k,r,\hat{K}} \; \leq \; \hat{C} \; \Big(|\hat{g}|_{k-1,p,\hat{K}} \; + \; |\hat{g}|_{k,p,\hat{K}} \Big) \; . \tag{5.54}$$

Using (5.53) and (5.54) in (5.52), we get

$$|\hat{E}_{\hat{K}}(\hat{g}(\hat{p} - \hat{Q}\hat{p}))| \; \leq \; \hat{C} \; \Big(|\hat{g}|_{k-1,p,\hat{K}} + |\hat{g}|_{k,p,\hat{K}} \Big) \; |\hat{p}|_{1,\hat{K}} \; . \tag{5.55}$$

Finally, combining (5.46),(5.51),(5.55) and observing that Theorem 4.3 infers

$$|\hat{g}|_{k-j,p,\hat{K}} \; \leq \; \hat{C} \; (\det(B_K))^{-1/p} \; h_K^{k-j} \; |g|_{k-j,p,K} \quad , \quad 0 \leq j \leq 1 \; ,$$

$$|\hat{p}|_{j,\hat{K}} \;\le\; \hat{C}\;(\det(B_K))^{-1/2}\; h_K^j\; |p|_{j,K} \quad , \quad 0 \le j \le 1 \;,$$

we arrive at the assertion. □

After having estimated both the approximation and the consistency errors, we are now in a position to establish an **a priori estimate of the global discretization error** in the $H^1(\Omega)$-norm.

For this purpose, we provide the following auxiliary result.

Lemma 5.3 Generalized Schwarz inequality for sums

Assume $a_i, b_i, c_i \in \mathbb{R}, 1 \le i \le m$, and $r_1, r_2, r_3 \in \mathbb{N}$ satisfying $\frac{1}{r_1} + \frac{1}{r_2} + \frac{1}{r_3} = 1$. Then, there holds

$$\sum_{i=1}^m |a_i b_i c_i| \;\le\; \Big(\sum_{i=1}^m |a_i|^{r_1}\Big)^{1/r_1} \Big(\sum_{i=1}^m |b_i|^{r_2}\Big)^{1/r_2} \Big(\sum_{i=1}^m |c_i|^{r_3}\Big)^{1/r_3} \;. \tag{5.56}$$

Proof. The proof is left as an exercise. □

Theorem 5.5 A priori estimate in the $H^1(\Omega)$-norm

In addition to the assumptions **(A1),(A2),(A3)**, suppose that for some integer $k \in \mathbb{N}$ we have

$$\hat{P} \;=\; P_k(\hat{K}) \;, \tag{5.57}$$

and that

$$H^{k+1}(\hat{K}) \;\to\; C^s(\hat{K}) \tag{5.58}$$

is a continuous imbedding, where $s \in \mathbb{N}_0$ is the maximal order of partial derivatives in the definition of the set $\hat{\Sigma}_{\hat{K}}$ of degrees of freedom. Moreover, let $\hat{Q}_{\hat{K}}$ be a quadrature formula with respect to the reference element $\hat{K}$ with positive weights $\hat{\omega}_{\ell,\hat{K}}, 1 \le \ell \le L$, satisfying

$$\hat{E}_{\hat{K}}(\hat{\varphi}) = 0 \;,\; \hat{\varphi} \in P_{2k-2}(\hat{K}) \;. \tag{5.59}$$

As far as the data of the model problem (5.9) are concerned, we require the existence of some $p \ge 2$ with $k > \frac{d}{p}$ such that the following **regularity assumptions** are met

$$a_{ij} \in W^{k,\infty}(\Omega) \;,\; 1 \le i,j \le d \quad , \quad f \in W^{k,p}(\Omega) \;. \tag{5.60}$$

We denote by $u \in V := H_0^1(\Omega)$ the solution of (5.9) and by $u_h \in V_h, h \in \mathcal{H}$ the solution of its finite element approximation (5.21), and we assume

$$u \in V \cap H^{k+1}(\Omega) \;. \tag{5.61}$$

Then, there exists a positive constant $C \in \mathbb{R}$, independent of $h \in \mathcal{H}$, such that the following **a priori error estimate** for the global discretization error holds true

(5.62)

$$\|u - u_h\|_{1,\Omega} \;\le\; C\, h^k \Big(|u|_{k+1,\Omega} \;+\; \sum_{i,j=1}^{d} \|a_{ij}\|_{k,\infty,\Omega}\; \|u\|_{k+1,\Omega} \;+\; \|f\|_{k,p,\Omega} \Big) \;.$$

Proof. The assumptions apply that the family $(a_h(\cdot,\cdot))_{h\in\mathcal{H}}$ of approximate bilinear forms $a_h(\cdot,\cdot) : V_h \times V_h \to \mathbb{R}, h \in \mathcal{H}$, is **uniformly V_h-elliptic**. We may thus apply **Strang's first lemma** and provide appropriate estimates of the **approximation error** and the **consistency error**.

(i) Approximation error

In view of the regularity assumption (5.61), it follows from Theorem 4.5 that

$$\|u - \Pi_h u\|_{1,\Omega} \;\le\; C\, h^k\; |u|_{k+1,\Omega} \;. \tag{5.63}$$

(ii) Consistency error, Part I

By Theorem 5.3 and the Cauchy-Schwarz inequality, for $w_h \in V_h$ we obtain

$$\begin{aligned} (5.64)\quad & |a(\Pi_h u, w_h) - a_h(\Pi_h u, w_h)| \;\le \\ & \le\; \sum_{K\in\mathcal{T}_h} \sum_{i,j=1}^{d} |E_K(a_{ij} \frac{\partial \Pi_K u}{\partial x_i} \frac{\partial w_h}{\partial x_j})| \;\le \\ & \le\; C \sum_{K\in\mathcal{T}_h} h_K^k \Big(\sum_{i,j=1}^{d} \|a_{ij}\|_{k,\infty,K} \Big)\; \|\Pi_K u\|_{k,K}\; |w_h|_{1,K} \;\le \\ & \le\; C\, h^k \Big(\sum_{i,j=1}^{d} \|a_{ij}\|_{k,\infty,K} \Big) \Big(\sum_{K\in\mathcal{T}_h} \|\Pi_K u\|_{k,K}^2 \Big)^{1/2} |w_h|_{1,\Omega} \;. \end{aligned}$$

Applying Theorem 4.5 once more, we get

$$\begin{aligned} (5.65)\quad & \Big(\sum_{K\in\mathcal{T}_h} \|\Pi_K u\|_{k,K}^2 \Big)^{1/2} \;\le \\ & \le\; \|u\|_{k,K} \;+\; \Big(\sum_{K\in\mathcal{T}_h} \|u - \Pi_K u\|_{k,K}^2 \Big)^{1/2} \;\le \\ & \le\; \|u\|_{k,K} \;+\; C\, h\; |u|_{k+1,\Omega} \;\le\; C\; \|u\|_{k+1,\Omega} \;. \end{aligned}$$

Using (5.65) in (5.64) yields

$$\sup_{w_h \in V_h} \frac{|a(\Pi_h u, w_h) - a_h(\Pi_h u, w_h)|}{\|w_h\|_{1,\Omega}} \leq \tag{5.66}$$

$$\leq C\ h^k \sum_{i,j=1}^{d} \|a_{ij}\|_{k,\infty,\Omega}\ \|u\|_{k+1,\Omega}\ . \tag{5.67}$$

(ii) Consistency error, Part II

Using Theorem 5.4, for $w_h \in V_h$ it follows that

$$(5.68)\qquad |\ell(w_h) - \ell_h(w_h)| \ \le \ \sum_{K\in\mathcal{T}_h} |E_K(fw_h)| \ \le$$
$$\le \ C \sum_{K\in\mathcal{T}_h} (\mathrm{meas}(K))^{(\frac{1}{2}-\frac{1}{p})} \ h_K^k \ \|f\|_{k,p,K} \ \|w_h\|_{1,K} \ .$$

Applying the **generalized Schwarz inequality** (Lemma 5.3) to the last sum in (5.68) with

$$m \ := \ \mathrm{card}(\mathcal{T}_h) \ , \ \frac{1}{r_1} \ := \ \frac{1}{2} - \frac{1}{p} \ , \ r_2 \ := \ p \ , \ r_3 \ := \ 2 \ ,$$

results in

$$(5.69)\qquad |\ell(w_h) - \ell_h(w_h)| \ \le \ \sum_{K\in\mathcal{T}_h} |E_K(fw_h)| \ \le$$
$$\le \ C \ h^k \ (\mathrm{meas}(\Omega))^{(\frac{1}{2}-\frac{1}{p})} \ \|f\|_{k,p,\Omega} \ \|w_h\|_{1,\Omega} \ .$$

We thus get

$$(5.70)\qquad \sup_{w_h\in V_h} \frac{|\ell(w_h) - \ell_h(w_h)|}{\|w_h\|_{1,\Omega}} \ \le \ C \ h^k \ (\mathrm{meas}(\Omega))^{(\frac{1}{2}-\frac{1}{p})} \ \|f\|_{k,p,\Omega} \ .$$

Finally, combining (5.63),(5.66) and (5.70), the assertion is a direct consequence of Strang's first lemma. □

5.2.2 Special quadrature formulas

We first consider **quadrature formulas** for **simplicial triangulations**.

Lemma 5.4 Quadrature formulas for simplicial triangulations

Let $\hat{K}$ be a non degenerate 2-simplex with vertices $\hat{a}_i, 1lei \le 3$, and denote by $\hat{a}_{ij}, 1 \le i < j \le 3$, the midpoints of the edges and by $\hat{a}_{123}$ the center of gravity. then, there holds:

(i) The quadrature formula

$$(5.71)\qquad \int_{\hat{K}} \hat{\varphi}(\hat{x}) \ d\hat{x} \ \approx \ \mathrm{meas}(\hat{K}) \ \hat{\varphi}(\hat{a}_{123})$$

is exact for polynomials $\hat{p} \in P_1(\hat{K})$, i.e.,

$$(5.72)\qquad \hat{E}(\hat{p}) \ = \ 0 \quad , \quad \hat{p} \in P_1(\hat{K}) \ .$$

(ii) The quadrature formula

$$\int_{\hat{K}} \hat{\varphi}(\hat{x})\, d\hat{x} \;\approx\; \frac{1}{3}\, \text{meas}(\hat{K}) \sum_{1 \le i < j \le 3} \hat{\varphi}(\hat{a}_{ij}) \tag{5.73}$$

is exact for polynomials $\hat{p} \in P_2(\hat{K})$, i.e.,

$$\hat{E}(\hat{p}) \;=\; 0 \quad , \quad \hat{p} \in P_2(\hat{K}) \;. \tag{5.74}$$

(iii) The quadrature formula
(5.75)

$$\int_{\hat{K}} \hat{\varphi}(\hat{x})\, d\hat{x} \;\approx\; \frac{1}{60}\, \text{meas}(\hat{K}) \Big(3 \sum_{i=1}^{3} \hat{\varphi}(\hat{a}_i) + 8 \sum_{1 \le i < j \le 3} \hat{\varphi}(\hat{a}_{ij}) + 27\, \hat{\varphi}(\hat{a}_{123}) \Big)$$

is exact for polynomials $\hat{p} \in P_3(\hat{K})$, i.e.,

$$\hat{E}(\hat{p}) \;=\; 0 \quad , \quad \hat{p} \in P_3(\hat{K}) \;. \tag{5.76}$$

Proof. Let $\lambda_i(\hat{x}), 1 \le i \le 3,$ be the barycentric coordinates of $\hat{x} \in \hat{K}$. Then, for $\alpha_i \in \mathbb{N}_0, 1 \le i \le 3,$ there holds

$$\int_{\hat{K}} \prod_{i=1}^{3} \lambda_i^{\alpha_i}(\hat{x})\, d\hat{x} \;=\; \text{meas}(\hat{K})\, \frac{d! \prod_{i=1}^{3} \alpha_i!}{(\sum_{i=1}^{3} \alpha_i + d)!} \;. \tag{5.77}$$

The proof of (5.77) is left as an exercise. □

We next consider **quadrature formulas** for **rectangular triangulations**.

Lemma 5.5 Quadrature formulas for rectangular triangulations

Let $\hat{K} := [0,1]^2$ with vertices

$$\hat{a}_1 := (0,0) \;,\; \hat{a}_2 := (1,0) \;,\; \hat{a}_3 := (1,1) \;,\; \hat{a}_4 := (0,1)$$

and center $\hat{a}_s := (1/2, 1/2)$. Then, there holds **(i)** The quadrature formula

$$\int_{\hat{K}} \hat{\varphi}(\hat{x})\, d\hat{x} \;\approx\; \frac{1}{4} \sum_{i=1}^{4} \hat{\varphi}(\hat{a}_i) \tag{5.78}$$

is exact for polynomials $\hat{p} \in Q_1(\hat{K})$, i.e.,

$$\hat{E}(\hat{p}) \;=\; 0 \quad , \quad \hat{p} \in Q_1(\hat{K}) \;. \tag{5.79}$$

(ii) The quadrature formula

$$\int_{\hat{K}} \hat{\varphi}(\hat{x})\ d\hat{x} \ \approx\ \hat{\varphi}(\hat{a}_s) \tag{5.80}$$

is exact for polynomials $\hat{p} \in Q_1(\hat{K})$, i.e.,

$$\hat{E}(\hat{p}) \ =\ 0 \quad , \quad \hat{p} \in Q_1(\hat{K}) \ . \tag{5.81}$$

(iii) Let $k \in \mathbb{N}_0$ and denote by $\hat{b}_i \in (0,1), 1 \le i \le k+1$, the **nodes** and by $\hat{\omega}_i, 1 \le i \le k+1$, the **weights** of the **Gauss-Legendre quadrature formula**

$$\int_0^1 \hat{\varphi}(\hat{x})\ d\hat{x} \ \approx\ \sum_{i=1}^{k+1} \hat{\omega}_i)\ \hat{\varphi}(\hat{b}_i) \ ,$$

which is exact for polynomials $\hat{p} \in P_{2k+1}(\hat{k}$.
Then, the quadrature formula

$$\int_{\hat{K}} \hat{\varphi}(\hat{x})\ d\hat{x} \ \approx \sum_{i_1,i_2 \in \{1,\ldots,k+1\}} \hat{\omega}_{i_1}\hat{\omega}_{i_2}\ \hat{\varphi}(\hat{b}_{i_1}, \hat{b}_{i_2}) \tag{5.82}$$

is exact for polynomials $\hat{p} \in Q_{2k+1}(\hat{K})$, i.e.,

$$\hat{E}(\hat{p}) \ =\ 0 \quad , \quad \hat{p} \in Q_{2k+1}(\hat{K}) \ . \tag{5.83}$$

Proof. The proof is left as an exercise. □

We note that the quadrature formulas (5.71),(5.73),(5.75) and (5.78), (5.80),(5.82) can be easily generalized to 3D.

5.3 Strang's second lemma

In this section, we will derive an **abstract error estimate**, known as **Strang's second lemma**, that is applicable for finite element approximations of variational equations

$$a(u,v) \ =\ \ell(v) \quad , \quad v \in V \ , \tag{5.84}$$

where the finite element spaces $V_h, h \in \mathcal{H}$, are no longer subspaces of the underlying infinite dimensional function space V.
We assume that $(V_h)_{h\in\mathcal{H}}$ is a family of finite dimensional linear spaces and suppose that $\|\cdot\|_h, h \in \mathcal{T}_h$, provides a **mesh-dependent norm** on $V_h + V$. We further suppose that the linear functional ℓ is well defined on $V_h + V$.
We consider an associated family of **approximate bilinear forms**

$$a_h(\cdot,\cdot) : (V_h + V) \times (V_h + V) \to \mathbb{R} \ . \tag{5.85}$$

We suppose that the family $(a_h(\cdot,\cdot))_{h\in\mathcal{H}}$ is **uniformly V_h-elliptic** in the sense that there exists a positive constant $\tilde{\alpha}$, independent of $h \in \mathcal{H}$, such that

$$a_h(v_h, v_h) \ \geq \ \tilde{\alpha}\ \|v_h\|^2_{V_h} \quad , \quad v_h \in V_h \ , \ h \in \mathcal{H} \ . \tag{5.86}$$

We further suppose that the family $(a_h(\cdot,\cdot))_{h\in\mathcal{H}}$ is **uniformly bounded** on $V_h + V$ in the sense that there exists a positive constant $\tilde{M}$, independent of $h \in \mathcal{H}$, such that

$$|a_h(u,v)| \ \leq \ \tilde{M}\ \|u\|_h\ \|v\|_h \quad , \quad u,v \in V_h + V \ , \ h \in \mathcal{H} \ . \tag{5.87}$$

We approximate (5.84) by the finite dimensional variational equations

$$a_h(u_h, v_h) \ = \ \ell(v_h) \quad , \quad v_h \in V_h \ , \ h \in \mathcal{H} \ . \tag{5.88}$$

Strang's second lemma may be viewed as another generalization of Céa's lemma.

Theorem 5.6 Strang's second lemma

Assume that $(a_h(\cdot,\cdot))_{h\in\mathcal{H}}$ is a **uniformly bounded** and **uniformly V_h-elliptic** family of approximate bilinear forms and suppose that $u \in V$ and $u_h \in V_h, h \in \mathcal{H}$, are the unique solutions of the variational equations (5.84) and (5.88), respectively.
Then, there exists a positive constant $C \in \mathbb{R}$, independent of $h \in \mathcal{H}$, such that

$$\|u - u_h\|_h \ \leq \ C \left(\inf_{v_h\in V_h} \|u - v_h\|_h + \sup_{w_h \in V_h} \frac{|a_h(u,w_h) - \ell(w_h)|}{\|w_h\|_h} \right) . \tag{5.89}$$

Proof. Taking advantage of the **uniform V_h-ellipticity** and observing (5.88), for an arbitrary $v_h \in V_h$ we obtain

$$\begin{aligned} &\tilde{\alpha}\ \|u_h - v_h\|_h^2 \ \leq \ a_h(u_h - v_h, u_h - v_h) \ = \\ &= \ a_h(u - v_h, u_h - v_h) \ + \ \Big(\ell(u_h - v_h) \ - \ a_h(u, u_h - v_h) \Big) \ . \end{aligned}$$

By the **uniform boundedness** (5.87) we get

$$\begin{aligned} &\tilde{\alpha}\ \|u_h - v_h\|_h \ \leq \ a_h(u_h - v_h, u_h - v_h) \ \leq \\ &\leq \ \tilde{M}\ \|u - v_h\|_h \ + \ \frac{|\ell(u_h - v_h) - a_h(u, u_h - v_h)|}{\|u_h - v_h\|_h} \ \leq \\ &\leq \ \tilde{M}\ \|u - v_h\|_h \ + \ \sup_{w_h\in V_h} \frac{|\ell(w_h) - a_h(u,w_h)|}{\|w_h\|_h} \ . \end{aligned}$$

In conjunction with the triangle inequality

$$\|u - u_h\|_h \ \leq \ \|u - v_h\|_h \ + \ \|u_h - v_h\|_h \ ,$$

the previous inequality gives the assertion. □

As in the case of Strang's first lemma, we see that the upper bound for the global discretization error consists of an **approximation error**

$$\inf_{v_h \in V_h} \|u - v_h\|_h \tag{5.90}$$

and the **consistency error**

$$\sup_{w_h \in V_h} \frac{|\ell(w_h) - a_h(u, w_h)|}{\|w_h\|_h} , \tag{5.91}$$

which can be estimated separately.

5.4 A priori error estimate for nonconforming finite element approximations

As an example for a **nonconforming finite element** in the discretization of second order elliptic boundary value problems with respect to a family of shape-regular simplicial triangulations $\mathcal{T}_h, h \in \mathcal{H}$, of the computational domain $\Omega \subset \mathbb{R}^d$ we consider the lowest order **Crouzeix-Raviart element** $CR_1(K) := (K, P_K, \Sigma_K), K \in \mathcal{T}_h$, also known as the **nonconforming P1-element**:

$$\begin{aligned} P_K &:= P_1(K) , \\ \Sigma_K &:= \{ \, p(m_i(K)) \, , \; 1 \le i \le d+1 \, \} \, , \; p \in P_K , \end{aligned} \tag{5.92}$$

where $m_i(K), 1 \le i \le d+1$, are the **midpoints** of the edges ($d = 2$) and the **centers of gravity** of the faces ($d = 3$), respectively (cf. Figure 5.1).

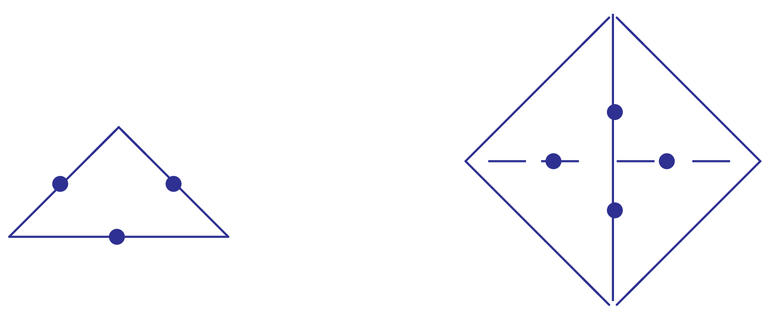

Fig. 5.1: Nonconforming Crouzeix-Raviart element

The associated **Crouzeix-Raviart finite element space** $CR_1(\Omega, \mathcal{T}_h)$, composed by the Crouzeix-Raviart elements $CR_1(K), K \in \mathcal{T}_h$, is then given by

$$CR_1(\Omega, \mathcal{T}_h) := \{ v_h \in L^2(\Omega) \mid v_h|_K \in P_1(K) \ , \ K \in \mathcal{T}_h \ , \ v_h \text{ is continuous in } m(f), f \in \mathcal{F}_h(\Omega) \} \ , \tag{5.93}$$

where $m(f), f \in \mathcal{F}_h(\Omega)$ are the midpoints (centers of gravity) of interior edges (faces).
Obviously, in general

$$CR_1(\Omega, \mathcal{T}_h) \not\subset H^1(\Omega) \ . \tag{5.94}$$

The subspace $CR_{1,0}(\Omega, \mathcal{T}_h)$ is defined by means of

$$CR_{1,0}(\Omega, \mathcal{T}_h) := \{ v_h \in CR_1(\Omega, \mathcal{T}_h) \mid v_h(m(f)) = 0 \ , \ f \in \mathcal{F}_h(\partial\Omega) \} \ .$$

We consider the approximation of **Poisson's equation** under **homogeneous Dirichlet boundary conditions**

$$a(u,v) := \int_\Omega \nabla u \cdot \nabla v dx = \ell(v) := \int_\Omega fv \ dx \ , \ v \in H_0^1(\Omega) \tag{5.95}$$

by

$$a_h(u_h, v_h) = \ell(v_h) \quad , \quad v_h \in CR_{1,0}(\Omega, \mathcal{T}_h) \ , \ h \in \mathcal{H} \ , \tag{5.96}$$

where the **mesh dependent bilinear form**

$$a_h(\cdot,\cdot) : H_0^1(\Omega) \oplus CR_{1,0}(\Omega, \mathcal{T}_h) \times H_0^1(\Omega) \oplus CR_{1,0}(\Omega, \mathcal{T}_h) \to \mathbb{R}$$

is given by

$$a_h(u,v) := \sum_{K \in \mathcal{T}_h(\Omega)} \int_K \nabla u \cdot \nabla v \ dx \ , \ u, v \in H_0^1(\Omega) \oplus CR_{1,0}(\Omega, \mathcal{T}_h) \ . \tag{5.97}$$

We associate with (5.97) the **mesh dependent norm**

$$\|v\|_h := \sqrt{a_h(v,v)} \quad , \quad v \in CR_{1,0}(\Omega, \mathcal{T}_h) \times H_0^1(\Omega) \ . \tag{5.98}$$

We will derive a **quasi-optimal a priori error estimate** in the $\| \cdot \|_h$-norm by an application of **Strang's second lemma**:

Theorem 5.7 A priori error estimate in the $\| \cdot \|_h$-norm
Let $u \in H_0^1(\Omega) \cap H^s(\Omega)$, $s = 2 \ (d = 2)$, $s = 3 \ (d = 3)$, be the solution of (5.95) and $u_h \in CR_{1,0}(\Omega, \mathcal{T}_h), h \in \mathcal{H}$, its nonconforming P1-approximations. Then, there exists a constant $C > 0$, depending only on the shape regularity of the triangulations $\mathcal{T}_h(\Omega), h \in \mathcal{H}$, such that

$$\|u - u_h\|_h \leq C \ h \ |u|_{s,\Omega} \ . \tag{5.99}$$

Proof. It is easily verified that the family $(a_h(\cdot,\cdot))_{h\in\mathcal{H}}$ of bilinear forms is **uniformly V_h-elliptic** ($V_h := CR_{1,0}(\Omega,\mathcal{T}_h)$). According to **Strang's second lemma** we have to estimate the **approximation error**

$$\inf_{v_h \in CR_{1,0}(\Omega,\mathcal{T}_h)} \|u - v_h\|_h \tag{5.100}$$

and the **consistency error**

$$\sup_{w_h \in CR_{1,0}(\Omega,\mathcal{T}_h)} \frac{a_h(u,w_h) - \ell(w_h)}{\|w_h\|_h} \ . \tag{5.101}$$

(i) Estimation of the approximation error

Since $S_{1,0}(\Omega,\mathcal{T}_h) \subset CR_{1,0}(\Omega,\mathcal{T}_h)$, in view of Theorem 4.7 we have

$$\inf_{v_h \in CR_{1,0}(\Omega,\mathcal{T}_h)} \|u - v_h\|_h \ \leq \inf_{v_h \in S_{1,0}(\Omega,\mathcal{T}_h)} \|u - v_h\|_h \ \leq \ C\, h\, |u|_{s,\Omega} \ . \tag{5.102}$$

(ii) Estimation of the consistency error

Observing that

$$-\ \Delta u(x) \ = \ f(x) \quad \text{f.a.a. } x \in \Omega \ ,$$

by **Green's formula** we obtain for $w_h \in CR_{1,0}(\Omega,\mathcal{T}_h)$

$$\begin{aligned} L_u(w_h) \ &:= \ a_h(u,w_h) \ - \ \ell(w_h) \ = \\ &= \sum_{K\in\mathcal{T}_h} \Big(\int_K \nabla u \cdot \nabla w_h \, dx \ - \ \int_K f w_h \, dx \Big) \ = \\ &= \sum_{K\in\mathcal{T}_h} \Big(\int_K \Big(-\ \Delta u \ - f \Big) w_h \, dx \ + \ \int_{\partial K} \frac{\partial u}{\partial \mathbf{n}} \, w_h \, d\sigma \Big) \ = \\ &= \sum_{K\in\mathcal{T}_h} \sum_{f\in\mathcal{K}} \int_f \frac{\partial u}{\partial \mathbf{n}} \, w_h \, d\sigma \ . \end{aligned} \tag{5.103}$$

Denoting by $w_h|_f$ the **integral mean**

$$w_h|_f \ := \ \frac{1}{\text{meas}(f)} \int_f w_h \, d\sigma \quad , \quad f \in \mathcal{F}_h(\Omega) \ ,$$

it follows that

$$L_u(w_h) \ = \ \sum_{K\in\mathcal{T}_h} \sum_{f\in\mathcal{K}} \int_f \frac{\partial u}{\partial \mathbf{n}} \, (w_h - w_h|_f) \, d\sigma \ . \tag{5.104}$$

Moreover, since $\frac{\partial I_h u}{\partial \mathbf{n}} \in P_0(K_\nu), 1 \leq \nu \leq 2$, we have

$$\int\limits_{f \cap K_\nu} \frac{\partial I_h u}{\partial \mathbf{n}} \; (w_h - w_h|_f) \; d\sigma \;=\; 0 \;,$$

and hence, (5.104) gives rise to

$$L_u(w_h) \;=\; \sum_{K \in \mathcal{T}_h} \sum_{f \in \mathcal{K}} \int\limits_f \frac{\partial (u - I_h u)}{\partial \mathbf{n}} \; (w_h - w_h|_f) \; d\sigma \;.$$

The Cauchy-Schwarz inequality yields

$$|L_u(w_h)| \;\leq\; \sum_{K \in \mathcal{T}_h} \sum_{f \in \mathcal{K}} \Bigl(\underbrace{\int\limits_f |\frac{\partial (u - I_h u)}{\partial \mathbf{n}}|^2 \; d\sigma}_{:= I_1} \; \underbrace{\int\limits_f |w_h - w_h|_f|^2 \; d\sigma}_{:= I_2} \Bigr)^{1/2} \;.$$

We will estimate I_1 and I_2 separately.

(ii)$_1$ Estimation of I_1

Taking advantage of the **affine equivalence** of the **Crouzeix-Raviart elements**, for the **reference element** $\hat{K}$ we find by the **trace theorem** and the **Bramble-Hilbert lemma**

$$\int\limits_{\partial \hat{K}} |\frac{\partial (u - I_h u)}{\partial \mathbf{n}}|^2 \; d\sigma \;\leq\; C \|u - I_h u\|^2_{2,\hat{K}} \;\leq\; C \; |u|^2_{s,\hat{K}} \;.$$

By a **standard scaling argument**

$$\int\limits_{\partial K} |\frac{\partial (u - I_h u)}{\partial \mathbf{n}}|^2 \; d\sigma \;\leq\; C \; h \; |u|^2_{s,K} \;. \tag{5.105}$$

(ii)$_2$ Estimation of I_2

Again, for the **reference element** $\hat{K}$ we find by the **Bramble-Hilbert lemma**

$$\int\limits_{\hat{f}} |\hat{w}_h - \hat{w}_h|_{\hat{f}}|^2 \; d\hat{\sigma} \;\leq\; C |\hat{w}_h|^2_{1,\hat{K}} \quad , \quad \hat{w}_h \in P_1(\hat{K}) \; , \; \hat{f} \in \mathcal{F}(\hat{K}) \; ,$$

and by a **standard scaling argument**, for $w_h \in CR_{1,0}(\Omega, \mathcal{T}_h)$ and $f \in \mathcal{F}(K), K \in \mathcal{T}_h$ we obtain

$$\int\limits_f |w_h - w_h|_f|^2 \; d\sigma \;\leq\; C \; h \; |w_h|^2_{1,K} \;. \tag{5.106}$$

Using (5.105) and (5.106), we finally obtain

$$|L_u(w_h)| \le 3\ C\ h \sum_{K \in \mathcal{T}_h} |u|_{s,K}\ |w_h|_{1,K} \le$$

$$\le C\ h \Big(\sum_{K \in \mathcal{T}_h} |u|^2_{s,K} \sum_{K \in \mathcal{T}_h} |w_h|^2_{1,K} \Big)^{1/2} = C\ h\ |u|_{s,\Omega}\ \|w_h\|_h\ . \quad \square$$