Chapter 8 Curl-conforming edge element methods

8.1 Maxwell's equations

8.1.1 Introduction

Electromagnetic phenomena can be described by the **electric field** $\mathbf{E}$, the **electric induction** $\mathbf{D}$, the **current density** $\mathbf{J}$ as well as the **magnetic field** $\mathbf{H}$ and the **magnetic induction** $\mathbf{B}$ according to **Maxwell's equations** given by **Faraday's law**

$$\frac{\partial \mathbf{B}}{\partial t} + \mathbf{curl}\,\mathbf{E} = 0 \quad \text{in } \mathbb{R}^3 , \tag{8.1}$$

where according to the **Gauss law**

$$\operatorname{div} \mathbf{B} = 0 \quad \text{in } \mathbb{R}^3 , \tag{8.2}$$

and **Ampère's law**

$$\frac{\partial \mathbf{D}}{\partial t} - \mathbf{curl}\,\mathbf{H} + \mathbf{J} = 0 \quad \text{in } \mathbb{R}^3 , \tag{8.3}$$

where, again observing the **Gauss law**

$$\operatorname{div} \mathbf{D} = \rho \quad \text{in } \mathbb{R}^3 . \tag{8.4}$$

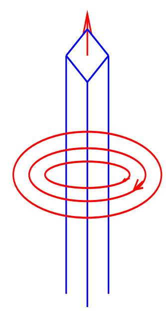

Figure 8.1: Faraday's law

In particular, Faraday's law describes how an electric field can be induced by a changing magnetic flux. It states that the induced electric field is proportional to the time rate of change of the magnetic flux through the circuit.
For $D \subset \mathbb{R}^3$ the **integral form of Faraday's law** states:

$$\int_D \frac{\partial \mathbf{B}}{\partial t}\, d\mathbf{x} = - \int_{\partial D} \mathbf{E} \wedge \mathbf{n}\, d\sigma \quad ,$$

where $\mathbf{E} \wedge \mathbf{n}$ is the tangential trace of $\mathbf{E}$ with $\mathbf{n}$ denoting the exterior unit normal. Observe the orientation of the induced electric field in Fig. 8.1. The Stokes' theorem implies

$$\int_{\partial D} \mathbf{E} \wedge \mathbf{n} \, d\sigma = \int_D \mathbf{curl}\ \mathbf{E} \, d\mathbf{x} \quad ,$$

and we thus obtain

$$\int_D \left[\frac{\partial \mathbf{B}}{\partial t} + \mathbf{curl}\ \mathbf{E} \right] d\mathbf{x} = 0 \quad .$$

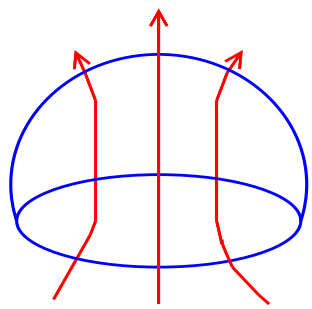

Figure 8.2: Gauss' law

For $D \subset \mathbb{R}^3$ the **integral form of the Gauss law of the magnetic field** states:

$$\int_{\partial D} \mathbf{n} \cdot \mathbf{B} \, d\sigma = 0 \quad ,$$

where $\mathbf{n} \cdot \mathbf{B}$ is the normal component of $\mathbf{B}$.
The Gauss' integral theorem implies

$$\int_{\partial D} \mathbf{n} \cdot \mathbf{B} \, d\sigma = \int_D \mathrm{div}\ \mathbf{B} \, d\mathbf{x} \quad ,$$

and we thus obtain $\mathrm{div}\mathbf{B} = 0$ as the differential form of the Gauss law. In other words, the Gauss law of the magnetic field says that the magnetic induction is a solenoidal vector field (source-free).

On the other hand, Ampère's law shows that an electric current can induce a magnetic field. It says that the path integral of the magnetic flux around a closed path is proportional to the electric current enclosed by the path.

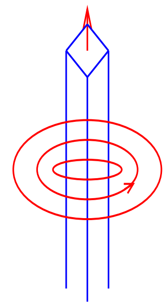

Figure 8.3: Ampére's law

For $D \subset \mathbb{R}^3$ the **integral form of Ampére's law** states:

$$\int_D (\frac{\partial \mathbf{D}}{\partial t} + \mathbf{J})\, d\mathbf{x} \;=\; \int_{\partial D} \mathbf{H} \wedge \mathbf{n}\, d\sigma \quad ,$$

where $\mathbf{H} \wedge \mathbf{n}$ is the tangential trace of $\mathbf{H}$.
The Stokes' theorem implies

$$\int_{\partial D} \mathbf{H} \wedge \mathbf{n}\, d\sigma \;=\; \int_D \mathbf{curl}\ \mathbf{H}\, d\mathbf{x} \quad ,$$

and we thus obtain

$$\int_D \left[\frac{\partial \mathbf{D}}{\partial t} - \mathbf{curl}\ \mathbf{H} + \mathbf{J}\right] d\mathbf{x} \;=\; 0 \quad ,$$

whence $\frac{\partial \mathbf{D}}{\partial t} - \mathbf{curl}\ \mathbf{H} + \mathbf{J} = 0$ as the differential form of the Ampére law.
Finally, the **Gauss law of the electric field** expresses the fact that the charges represent the source of the electric induction $\mathbf{D}$.

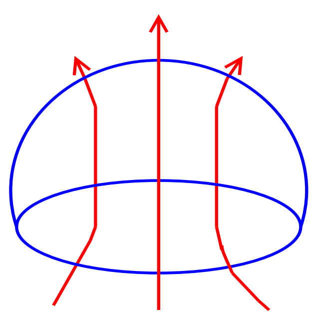

Figure 8.4: Gauss' law of the electric field

For $D \subset \mathbb{R}^3$ the **integral form of the Gauss law of the electric field** states:

$$\rho \int_{\partial D} \mathbf{n} \cdot \mathbf{D} \, d\sigma \; = \; \int_D \rho \, d\mathbf{x} \quad ,$$

where $\mathbf{n} \cdot \mathbf{D}$ is the normal component of $\mathbf{D}$.
The Gauss' integral theorem implies

$$\int_{\partial D} \mathbf{n} \cdot \mathbf{D} \, d\sigma \; = \; \int_D \operatorname{div} \mathbf{D} \, d\mathbf{x} \quad ,$$

from which we deduce $\operatorname{div} \mathbf{D} = \rho$ as the differential form of the Gauss law.
The fields $\mathbf{D}$, $\mathbf{E}$, $\mathbf{J}$, and $\mathbf{B}$, $\mathbf{H}$ are related by the **material laws**

$$\mathbf{D} \; = \; \varepsilon \, \mathbf{E} \, + \, \mathbf{P} \quad , \tag{8.5}$$

$$\mathbf{J} \; = \; \sigma \, \mathbf{E} \, + \, \mathbf{J}_e \quad , \tag{8.6}$$

$$\mathbf{B} \; = \; \mu \, \mathbf{H} \, + \, \mu_0 \, \mathbf{M} \quad , \tag{8.7}$$

where $\mathbf{J}_e$, $\mathbf{M}$, and $\mathbf{P}$ are the **impressed current density**, **magnetization**, and **electric polarization**, respectively.
Here, $\varepsilon = \varepsilon_r \varepsilon_0$ and $\mu = \mu_r \mu_0$ are the **electric permittivity** and **magnetic permeability** of the medium with ε_0 and μ_0 denoting the permittivity and permeability of the vacuum (ε_r and μ_r are referred to as the **relative permittivity** and the **relative permeability**).

At interfaces $\Gamma := \bar{\Omega}_1 \cap \bar{\Omega}_2$, $\Omega_1 \cap \Omega_2 = \emptyset$, separating different media Ω_1 , $\Omega_2 \subset \mathbb{R}^3$, **transmission conditions** have to be satisfied.
According to (8.4), the sources of the electric field are given by the electric charges. Denoting by $\mathbf{n}$ the unit normal on Γ pointing into the direction of Ω_2, the normal component $\mathbf{n} \cdot \mathbf{D}$ of the electric induction experiences a jump

$$[\mathbf{n} \cdot \mathbf{D}]_\Gamma \; := \; \mathbf{n} \cdot (\mathbf{D}|_{\Gamma \cap \bar{\Omega}_2} \; - \; \mathbf{D}|_{\Gamma \cap \bar{\Omega}_1}) \; = \; \eta \tag{8.8}$$

with η denoting the **surface charge**.
On the other hand, the tangential trace $\mathbf{E} \wedge \mathbf{n}$ of the electric field behaves continuously

$$[\mathbf{E} \wedge \mathbf{n}]_\Gamma \; := \; (\mathbf{E}|_{\Gamma \cap \bar{\Omega}_2} \; - \; \mathbf{E}|_{\Gamma \cap \bar{\Omega}_1}) \wedge \mathbf{n} = \; 0 \; , \tag{8.9}$$

which is in accordance with the physical laws, since otherwise a nonzero jump would indicate the existence of a magnetic surface current.
Since the magnetic induction $\mathbf{B}$ is solenoidal, the normal component

$\mathbf{n} \cdot \mathbf{B}$ must behave continuously, i.e.,

$$[\mathbf{n} \cdot \mathbf{B}]_\Gamma = 0 , \tag{8.10}$$

whereas the tangential trace $\mathbf{H} \wedge \mathbf{n}$ of the magnetic field undergoes a jump according to

$$[\mathbf{H} \wedge \mathbf{n}]_\Gamma = \mathbf{j}_\Gamma , \tag{8.11}$$

where $\mathbf{j}_\Gamma$ is the **surface current**.
Finally, the continuity of the current at interfaces requires that the normal component $\mathbf{n} \cdot \mathbf{J}$ of the current density satisfies

$$[\mathbf{n} \cdot \mathbf{J}]_\Gamma = - \frac{\partial \eta}{\partial t} . \tag{8.12}$$

8.1.2 Electromagnetic Potentials

The special form of Maxwell's equations (8.1)-(8.7) allows to introduce **electromagnetic potentials** which facilitate the computation of electromagnetic field problems by reducing the number of unknowns.

(i) Electric Scalar Potential

In case of an **electrostatic field** in a medium occupying a bounded simply-connected domain $\Omega \subset \mathbb{R}^3$, Faraday's law (8.1) reduces to

$$\mathbf{curl}\ \mathbf{E} = 0 \qquad \text{in } \Omega . \tag{8.13}$$

Consequently, the electric field $\mathbf{E}$ can be represented as the gradient of an **electric scalar potential** φ according to

$$\mathbf{E} = - \mathbf{grad}\ \varphi . \tag{8.14}$$

(ii) Magnetic Vector Potential

The solenoidal character of the magnetic induction $\mathbf{B}$ according to (8.2) implies the existence of a **magnetic vector potential** $\mathbf{A}$ such that

$$\mathbf{B} = \mathbf{curl}\ \mathbf{A} . \tag{8.15}$$

Moreover, the material law (8.7) gives

$$\mathbf{H} = \mu^{-1}\ \mathbf{B} - \mu_r^{-1}\ \mathbf{M} ,$$

and hence,

$$\mathbf{H} = \mu^{-1}\ \mathbf{curl}\ \mathbf{A} - \mu_r^{-1}\ \mathbf{M} . \tag{8.16}$$

Replacing $\mathbf{H}$ in Ampère's law (8.3) by (8.16), we obtain

$$\mathbf{curl}\ (\mu^{-1}\ \mathbf{curl}\ \mathbf{A} - \mu_r^{-1}\ \mathbf{M}) = \mathbf{J} + \frac{\partial \mathbf{D}}{\partial t} . \tag{8.17}$$

On the other hand, replacing $\mathbf{B}$ in Faraday's law (8.1) by (8.15), we get

$$\mathbf{curl}\ (\mathbf{E} + \frac{\partial \mathbf{A}}{\partial t}) = 0 . \tag{8.18}$$

From (8.18) we deduce the existence of an **electric scalar potential** φ such that

$$\mathbf{E} = - \frac{\partial \mathbf{A}}{\partial t} - \mathbf{grad}\ \varphi . \tag{8.19}$$

Using (8.19) in (8.17) and observing the material laws (8.5) (with $\mathbf{P} = 0$) and (8.6), we arrive at the following wave-type equation for the magnetic vector potential $\mathbf{A}$:

$$\varepsilon \frac{\partial^2 \mathbf{A}}{\partial t^2} + \sigma \frac{\partial \mathbf{A}}{\partial t} + \mathbf{curl}\ (\mu^{-1}\ \mathbf{curl}\ \mathbf{A}) = \tag{8.20}$$
$$= \mathbf{J}_e + \mathbf{curl}\ \mu_r^{-1}\ \mathbf{M} - \sigma\ \mathbf{grad}\ \varphi - \varepsilon \frac{\partial}{\partial t} (\mathbf{grad}\ \varphi)$$

Since the **curl**-operator has a nontrivial kernel, the magnetic vector potential $\mathbf{A}$ is not uniquely determined by (8.15). This can be taken care of by a proper **gauging** which specifies the divergence of the potential. We distinguish between the **Coulomb gauge** given by

$$\mathrm{div}\ \mathbf{A} = 0 \tag{8.21}$$

and the **Lorentz gauge**

$$\Delta \varphi = - \frac{\partial \mathrm{div} \mathbf{A}}{\partial t} , \tag{8.22}$$

which is widely used in electromagnetic wave propagation problems.

(iii) Magnetic Scalar Potential

In case of a magnetostatic problem without currents, i.e., $\mathbf{J} = 0, \mathbf{D} = 0$, and vanishing magnetization $\mathbf{M} = 0$, equation (8.3) reduces to

$$\mathbf{curl}\ \mathbf{H} = 0 . \tag{8.23}$$

As for electrostatic problems, (8.23) implies the existence of a **magnetic scalar potential** ψ such that

$$\mathbf{H} = - \mathbf{grad}\ \psi . \tag{8.24}$$

The solenoidal character of the magnetic induction (8.2) and the material law (8.7) imply, that ψ satisfies the elliptic differential equation

$$\mathrm{div}\ (\mu\ \mathbf{grad}\ \psi) = 0 . \tag{8.25}$$

We further note that even problems with nonzero current density $\mathbf{J}$ can be cast in terms of the magnetic scalar potential. For this purpose, we decompose the magnetic field $\mathbf{H}$ according to

$$\mathbf{H} = \mathbf{H}_1 + \mathbf{H}_2 \tag{8.26}$$

into an irrotational part $\mathbf{H}_1$, i.e., $\mathbf{curl}\ \mathbf{H}_1 = 0$, and a second part $\mathbf{H}_2$ that can be computed by means of the **Bio-Savart law**

$$\mathbf{H}_2 = \frac{1}{4\pi} \int_\Omega \frac{\mathbf{J} \wedge \mathbf{x}}{|\mathbf{x}|^3}\, d\mathbf{x} . \tag{8.27}$$

In this case, we have

$$\mathbf{H}_1 = - \mathbf{grad}\ \psi_R , \tag{8.28}$$

with ψ_R being referred to as the **reduced magnetic scalar potential**. It follows readily that ψ_R satisfies the elliptic differential equation

$$\mathrm{div}\ (\mu\ \mathbf{grad}\ \psi_R) = \mathrm{div}\ \mu\ \mathbf{H}_2 . \tag{8.29}$$

8.1.3 Electrostatic Problems

In case of electrostatic problems, according to (8.14) the electric field $\mathbf{E}$ is given by the gradient of an electric scalar potential φ. Using (8.4) as well as the material law (8.5), we arrive at the following linear second order elliptic differential equation

$$- \mathrm{div}\ \varepsilon\ \mathbf{grad}\ \varphi = \rho - \mathrm{div}\ \mathbf{P} \quad \text{in } \Omega . \tag{8.30}$$

On the boundary $\Gamma_1 \subset \partial\Omega$, where the normal component $\mathbf{n} \cdot \mathbf{D}$ of the electric induction $\mathbf{D}$ is given by means of a prescribed surface current η, we obtain the **Neumann boundary condition**

$$\mathbf{n} \cdot \varepsilon\ \mathbf{grad}\ \varphi = \eta + \mathbf{n} \cdot \mathbf{P} \quad \text{on } \Gamma_1 . \tag{8.31}$$

On the other hand, if the boundary $\Gamma_2 \subset \partial\Omega\ ,\ \Gamma_2 \cap \Gamma_1 = \emptyset$, only contains metallic contacts, the electric field is perpendicular to Γ_2. In other words, the tangential trace $\mathbf{n} \wedge \mathbf{E}$ vanishes, and we get

$$\mathbf{E} \wedge \mathbf{n} = - \mathbf{grad}\ \varphi \wedge \mathbf{n} = 0 \quad \text{on } \Gamma_2 ,$$

from which we deduce the **Dirichlet boundary condition**

$$\varphi = g \quad \text{on } \Gamma_2 , \tag{8.32}$$

where the constant g is given by prescribed voltages.

The variational formulation of (8.30),(8.31),(8.32) involves the Hilbert space

$$H^1_{g,\Gamma_2}(\Omega) := \{ v \in H^1(\Omega) \mid v|_{\Gamma_2} = g \} \tag{8.33}$$

and can be derived as follows:
Multiplying (8.30) by $v \in H^1_{0,\Gamma_2}(\Omega)$ and integrating over Ω, Green's formula implies

$$
\begin{aligned}
(8.34)\qquad & - \int_\Omega \mathrm{div}\, \varepsilon\, \mathbf{grad}\, \varphi\ v\ d\mathbf{x} \ = \\
& = \int_\Omega \varepsilon\, \mathbf{grad}\, \varphi \cdot \mathbf{grad}\, v\ d\mathbf{x} \ - \ \int_\Gamma \mathbf{n} \cdot \varepsilon\, \mathbf{grad}\, \varphi\ v\ d\sigma \ = \\
& = \int_\Omega \varepsilon\, \mathbf{grad}\, \varphi \cdot \mathbf{grad}\, v\ d\mathbf{x} \ - \ \int_{\Gamma_1} (\eta \ + \ \mathbf{n} \cdot \mathbf{P})\ v\ d\sigma \ .
\end{aligned}
$$

Moreover, applying Green's formula once more, we have

$$
\begin{aligned}
(8.35)\qquad & - \int_\Omega \mathrm{div}\, \mathbf{P}\ v\ d\mathbf{x} \ = \\
& = \int_\Omega \mathbf{P} \cdot \mathbf{grad}\, v\ d\mathbf{x} \ - \ \int_\Gamma \mathbf{n} \cdot \mathbf{P}\ v\ d\sigma \ = \\
& = \int_\Omega \mathbf{P} \cdot \mathbf{grad}\, v\ d\mathbf{x} \ - \ \int_{\Gamma_1} \mathbf{n} \cdot \mathbf{P}\ v\ d\sigma \ .
\end{aligned}
$$

Using (8.34) and (8.35), the variational problem reads:
Find $\varphi \in H^1_{g,\Gamma_2}(\Omega)$ such that

$$(8.36)\qquad a(\varphi, v) \ = \ \ell(v) \quad , \quad v \in H^1_{0,\Gamma_2}(\Omega) \ ,$$

where $a(\cdot,\cdot)$ is the bilinear form

$$(8.37)\qquad a(\varphi, v) \ = \ \int_\Omega \varepsilon\, \mathbf{grad}\, \varphi \cdot \mathbf{grad}\, v\ d\mathbf{x}$$

and the functional $\ell(\cdot)$ is given by

$$(8.38)\ \ell(v) \ = \ \int_\Omega \rho\ v\ d\mathbf{x} \ + \ \int_\Omega \mathbf{P} \cdot \mathbf{grad}\, v\ d\mathbf{x} \ + \ \int_{\Gamma_1} \eta\ v\ d\sigma \ .$$

8.1.4 Magnetostatic Problems

For **magnetostatic problems** we use the potentials $\mathbf{A}$ and φ according to (8.15) and (8.19). Equation (8.20) reduces to

$$(8.39)\qquad \mathbf{curl}\ (\mu^{-1}\ \mathbf{curl}\ \mathbf{A} \ - \ \mu_r^{-1}\ \mathbf{M}) \ = \ \mathbf{J}_e \ - \ \sigma\ \mathbf{grad}\ \varphi \ =: \ \mathbf{f} \ .$$

As far as boundary conditions are concerned, we assume $\partial\Omega = \Gamma_1 \cup \Gamma_2$, $\Gamma_1 \cap \Gamma_2 = \emptyset$. On Γ_1 we assume vanishing tangential trace of $\mathbf{A}$

$$\mathbf{A} \wedge \mathbf{n} \ = \ 0 \quad \text{on } \Gamma_1 \ , \tag{8.40}$$

whereas on Γ_2 we suppose that

$$\mathbf{H} \wedge \mathbf{n} \ = \ \mathbf{j}_{\Gamma_2} \quad \text{on } \Gamma_2 \ ,$$

where $\mathbf{j}_{\Gamma_2}$ is the surface current density on Γ_2. Taking (8.16) into account, we get

$$(\mu^{-1} \ \mathbf{curl} \ \mathbf{A} \ - \ \mu_r^{-1} \ \mathbf{M}) \wedge \mathbf{n} \ = \ \mathbf{j}_{\Gamma_2} \quad \text{on } \Gamma_2 \ . \tag{8.41}$$

8.1.5 The Eddy Currents Equations

The **eddy current equations** represent the quasi-stationary limit of Maxwell's equations and describe the low frequency regime characterized by slowly time varying processes in conductive media. In this case, we have

$$\sigma \ \mathbf{E} \ \gg \ \frac{\partial \varepsilon \mathbf{E}}{\partial t} \ , \tag{8.42}$$

which means that the dielectric displacement can be neglected.
Hence, (8.20) reduces to the parabolic type equation

$$\sigma \ \frac{\partial \mathbf{A}}{\partial t} \ + \ \mathbf{curl} \ (\mu^{-1} \ \mathbf{curl} \ \mathbf{A} \ - \ \mu_r^{-1} \ \mathbf{M}) \ = \ \mathbf{J}_e \ - \ \sigma \ \mathbf{grad} \ \varphi \ . \tag{8.43}$$

8.1.6 The Time-Harmonic Maxwell Equations

We consider a homogeneous, nonconducting medium (i.e., $\sigma = 0$ and $\mathbf{J}_e = \mathbf{M} = \mathbf{P} = 0$) with electric permittivity ε and magnetic permeability μ. In this case, Maxwell's equations (8.1),(8.3) reduce to

$$\varepsilon \ \frac{\partial \mathbf{E}}{\partial t} \ - \ \mathbf{curl} \ \mathbf{H} \ = \ 0 \ , \tag{8.44}$$

$$\mu \ \frac{\partial \mathbf{H}}{\partial t} \ + \ \mathbf{curl} \ \mathbf{E} \ = \ 0 \ . \tag{8.45}$$

Applying the divergence to both equations, we see that

$$\frac{\partial}{\partial t} \ \text{div} \ \mathbf{E}(\mathbf{x},t) \ = \ \frac{\partial}{\partial t} \ \text{div} \ \mathbf{H}(\mathbf{x},t) \ = \ 0 \ , \tag{8.46}$$

which implies

$$\text{div} \ \mathbf{E}(\mathbf{x},t) \ = \ \text{div} \ \mathbf{H}(\mathbf{x},t) \ = \ 0 \ , \tag{8.47}$$

provided $\operatorname{div} \mathbf{E}(\mathbf{x},t_0) = \operatorname{div} \mathbf{H}(\mathbf{x},t_0) = 0$ at initial time t_0.
Differentiating (8.44),(8.45) with respect to time, we get

$$\frac{\partial^2 \mathbf{E}}{\partial t^2} - \frac{1}{\varepsilon} \mathbf{curl} \frac{\partial \mathbf{H}}{\partial t} = 0 , \tag{8.48}$$

$$\frac{\partial^2 \mathbf{H}}{\partial t^2} + \frac{1}{\mu} \mathbf{curl} \frac{\partial \mathbf{E}}{\partial t} = 0 . \tag{8.49}$$

Replacing $\frac{\partial \mathbf{H}}{\partial t}$ in (8.48) by (8.45) and $\frac{\partial \mathbf{E}}{\partial t}$ in (8.49) by (8.44), we obtain

$$\frac{\partial^2 \mathbf{E}}{\partial t^2} + \frac{1}{\varepsilon\mu} \mathbf{curl}\ \mathbf{curl}\ \mathbf{E} = 0 , \tag{8.50}$$

$$\frac{\partial^2 \mathbf{H}}{\partial t^2} + \frac{1}{\varepsilon\mu} \mathbf{curl}\ \mathbf{curl}\ \mathbf{H} = 0 . \tag{8.51}$$

Taking (8.47) into account and observing the vectorial identity

$$\Delta \mathbf{E} = \mathbf{grad} \operatorname{div} \mathbf{E} - \mathbf{curl}\ \mathbf{curl}\ \mathbf{E} ,$$

we finally see that $\mathbf{E}$ and $\mathbf{H}$ are solutions of the wave equations

$$\frac{\partial^2 \mathbf{E}}{\partial t^2} - c^2 \Delta \mathbf{E} = 0 , \tag{8.52}$$

$$\frac{\partial^2 \mathbf{H}}{\partial t^2} - c^2 \Delta \mathbf{H} = 0 , \tag{8.53}$$

where the **speed of light** in the medium is given by

$$c = \frac{1}{\sqrt{\varepsilon\mu}} . \tag{8.54}$$

The time-harmonic solutions of Maxwell equations, also called **plane waves**, are complex-valued fields

$$\begin{aligned} \mathbf{E}(\mathbf{x},t) &= \operatorname{Re} \big(\mathbf{E}(\mathbf{x}) \exp(-i\omega t) \big) , \\ \mathbf{H}(\mathbf{x},t) &= \operatorname{Re} \big(\mathbf{H}(\mathbf{x}) \exp(-i\omega t) \big) \end{aligned} \tag{8.55}$$

that satisfy the system of **time-harmonic Maxwell equations**

$$\begin{aligned} \mathbf{curl}\ \mathbf{H} + i\omega\varepsilon \mathbf{E} &= 0 , \\ \mathbf{curl}\ \mathbf{E} - i\omega\mu \mathbf{H} &= 0 , \end{aligned} \tag{8.56}$$

where ω stands for the frequency of the electromagnetic waves.
Similar computations as done before reveal that $\mathbf{E}$ and $\mathbf{H}$ satisfy (8.47) and thus the **vectorial Helmholtz equations**

$$\Delta \mathbf{E} + k^2 \mathbf{E} = 0 , \tag{8.57}$$

$$\Delta \mathbf{E} + k^2 \mathbf{E} = 0 , \tag{8.58}$$

where $k = \omega \sqrt{\varepsilon\mu}$ is the **wave number**.

8.2 The space $\mathbf{H}(\mathbf{curl}, \Omega)$ and its trace spaces

8.2.1 The space $\mathbf{H}(\mathbf{curl}, \Omega)$

Let $\Omega \subset \mathbb{R}^3$ be a simply connected polyhedral domain with boundary $\Gamma = \partial\Omega$ which can be split into N open faces $\Gamma_i, 1 \leq i \leq N$, such that $\Gamma = \cup_{i=1}^N \bar{\Gamma}_i$.
A generic point $\mathbf{x} \in \Omega$ is given coordinate-wise by $\mathbf{x} = (x_1, x_2, x_3)^T$. We refer to $\mathbf{n}$ as the unit outward normal to Γ and set $\mathbf{n}_i := \mathbf{n}|_{\Gamma_i}, 1 \leq i \leq N$. Moreover, we denote by $e_{ij} := \bar{\Gamma}_i \cap \bar{\Gamma}_j \neq \emptyset$ the common edge of two adjacent faces $\Gamma_i, \Gamma_j \subset \Gamma, 1 \leq i \neq j \leq N$, and to $\mathbf{t}_{ij}$ as a unit vector parallel to e_{ij}. We further set $\mathbf{t}_i := \mathbf{t}_{ij} \wedge \mathbf{n}_i$. The couple $(\mathbf{t}_i, \mathbf{t}_{ij})$ is an orthonormal basis of the plane generated by Γ_i.
We denote by $\mathcal{D}(\Omega)$ the space of all infinitely often differentiable functions with compact support in Ω and by $\mathcal{D}'(\Omega)$ its dual space referring to $< \cdot, \cdot >$ as the dual pairing between $\mathcal{D}'(\Omega)$ and $\mathcal{D}(\Omega)$.
For $\varphi \in \mathcal{D}(\Omega)$ we refer to **grad** $\varphi = \nabla\varphi$ as the gradient operator

$$\nabla\varphi \;:=\; (\frac{\partial\varphi}{\partial x_1}, \frac{\partial\varphi}{\partial x_2}, \frac{\partial\varphi}{\partial x_3})^T \; .$$

Further, for $\mathbf{q} = (q_1, q_2, q_3)^T \in \mathcal{D}(\Omega)^3$ we denote by $\mathbf{curl}\;\mathbf{q} = \nabla \wedge \mathbf{q}$ the rotation of $\mathbf{q}$

$$\nabla \wedge \mathbf{q} \;:=\; \begin{pmatrix} \frac{\partial q_3}{\partial x_2} - \frac{\partial q_2}{\partial x_3} \\ \frac{\partial q_1}{\partial x_3} - \frac{\partial q_3}{\partial x_1} \\ \frac{\partial q_2}{\partial x_1} - \frac{\partial q_1}{\partial x_2} \end{pmatrix} \; .$$

By taking advantage of distributional derivatives, we are allowed to define the operators **curl** on $L^2(\Omega)^3$:
Given $\mathbf{j} \in L^2(\Omega)^3$, we define $\mathbf{curl}\;\mathbf{j} \in \mathcal{D}'(\Omega)^3$ by means of

$$< \mathbf{curl}\;\mathbf{j}, \boldsymbol{\varphi} > = \int_\Omega \mathbf{j} \cdot \mathbf{curl}\;\boldsymbol{\varphi}\; d\mathbf{x} \quad , \quad \boldsymbol{\varphi} \in \mathcal{D}(\Omega)^3 \; .$$

Definition 8.1 The space $\mathbf{H}(\mathbf{curl}, \Omega)$

The space $\mathbf{H}(\mathbf{curl}, \Omega)$ is defined by

$$\mathbf{H}(\mathbf{curl}; \Omega) \;:=\; \{ \; \mathbf{q} \in L^2(\Omega)^3 \mid \mathbf{curl}\;\mathbf{q} \in L^2(\Omega)^3 \; \} \; . \tag{8.59}$$

It is a Hilbert space with respect to the inner product

$$\begin{aligned} (\mathbf{j}, \mathbf{q})_{curl,\Omega} \;&:= \\ &:= \; (\mathbf{j}, \mathbf{q})_{0,\Omega} \;+\; (\mathbf{curl}\;\mathbf{j}, \mathbf{curl}\;\mathbf{q})_{0,\Omega} \quad , \quad \mathbf{j}, \mathbf{q} \in \mathbf{H}(\mathbf{curl}; \Omega) \; . \end{aligned} \tag{8.60}$$

The associated norm will be denoted by $\| \cdot \|_{curl,\Omega}$.

8.2.2 Traces, trace mappings, and trace spaces I

We set $\mathcal{D}(\bar{\Omega}) := \{\varphi|_\Omega \mid \varphi \in \mathcal{D}(\mathbb{R}^3)\}$. For vector fields $\mathbf{q} \in \mathcal{D}(\bar{\Omega})^3$ we define the **tangential trace mapping**

(8.61)
$$\gamma_{\mathbf{t}} \quad := \quad \mathbf{q} \wedge \mathbf{n}|_\Gamma \ .$$

Further, we consider the **tangential components trace mapping**

(8.62)
$$\pi_{\mathbf{t}} \quad := \quad \mathbf{n} \wedge (\mathbf{q} \wedge \mathbf{n})|_\Gamma \ .$$

Recalling that $\mathcal{D}(\bar{\Omega})^3$ is dense in $H^1(\Omega)^3$, it is easy to see that the mappings $\gamma_{\mathbf{t}}$ and $\pi_{\mathbf{t}}$ can be extended to linear continuous mappings from $H^1(\Omega)^3$ into $\mathbf{H}_-^{1/2}(\Gamma)$.
However, the range of $\gamma_{\mathbf{t}}$ and the range of $\pi_{\mathbf{t}}$ are proper subspaces of $\mathbf{H}_-^{1/2}(\Gamma)$, as will be shown next. For this purpose, we need the following characterization of $H^{1/2}(\Gamma)$ (cf., e.g., [?]; Thm. 1.5):

Theorem 8.1 Characterization of $H^{1/2}(\Gamma)$

A function φ belongs to $H^{1/2}(\Gamma)$ if and only if $\varphi|_{\Gamma_i} \in H^{1/2}(\Gamma_i)$, $1 \leq i \leq N$, and

(8.63)
$$\int_{\Gamma_i} \int_{\Gamma_j} \frac{|\varphi(\mathbf{x}) \ - \ \varphi(\mathbf{y})|^2}{|\mathbf{x} \ - \ \mathbf{y}|^3} \, d\sigma(\mathbf{x}) \, d\sigma(\mathbf{y}) \ < \ \infty$$

for all $1 \leq i \neq j \leq N$ such that $\bar{\Gamma}_i \cap \bar{\Gamma}_j = e_{ij} \neq \emptyset$.

Definition 8.2 Equality on common edges of faces

Assume $\Gamma_i, \Gamma_j \subset \Gamma$, $i \neq j$ such that $e_{ij} = \bar{\Gamma}_i \cap \bar{\Gamma}_j \neq \emptyset$, and let $\varphi_i \in H^{1/2}(\Gamma_i)$ and $\varphi_j \in H^{1/2}(\Gamma_j)$. We define equality on e_{ij} by means of

(8.64)
$$\varphi_i \ =_{e_{ij}} \varphi_j \Longleftrightarrow \int_{\Gamma_i} \int_{\Gamma_j} \frac{|\varphi(\mathbf{x}) \ - \ \varphi(\mathbf{y})|^2}{|\mathbf{x} \ - \ \mathbf{y}|^3} \, d\sigma(\mathbf{x}) \, d\sigma(\mathbf{y}) \ < \ \infty \ .$$

We further introduce the set of indices

$$\mathcal{I}_i \ := \ \{j \in \{1, ..., N\} \mid \bar{\Gamma}_i \cap \bar{\Gamma}_j \ = \ e_{ij} \ \neq \ \emptyset\}$$

and define the space

(8.65)
$$\mathbf{H}_{||}^{1/2}(\Gamma) \ := \ \{\mathbf{q} \in \mathbf{H}_-^{1/2}(\Gamma) \mid \mathbf{t}_{ij} \cdot \mathbf{q}_i \ =_{e_{ij}} \ \mathbf{t}_{ij} \cdot \mathbf{q}_j \ , \ 1 \leq i \leq N \ , \ j \in \mathcal{I}_i\} \ .$$

Lemma 8.1 The space $\mathbf{H}_{||}^{1/2}(\Gamma)$

The space $\mathbf{H}_{||}^{1/2}(\Gamma)$ is a Hilbert space with respect to the norm

$$\|\mathbf{q}\|_{||,1/2,\Gamma} := \sum_{i=1}^{N} \|\mathbf{q}_i\|_{1/2,\Gamma_i}^2 + \sum_{i=1}^{N} \sum_{j\in\mathcal{I}_i} \int_{\Gamma_i} \int_{\Gamma_j} \frac{|\mathbf{t}_{ij}\cdot\mathbf{q}_i(\mathbf{x}) - \mathbf{t}_{ij}\cdot\mathbf{q}_j(\mathbf{y})|^2}{|\mathbf{x}-\mathbf{y}|^3}\, d\sigma(\mathbf{x})\, d\sigma(\mathbf{y}).$$

Proof. Let $(\mathbf{q}^k)_{k\in\mathbb{N}} \subset \mathbf{H}_{||}^{1/2}(\Gamma)$ be a Cauchy sequence with respect to $\|\cdot\|_{||,1/2,\Gamma}$. Obviously, there exists $\mathbf{q} \in \mathbf{H}_{-}^{1/2}(\Gamma)$ such that $\mathbf{q}^k \to \mathbf{q}$ as $k \to \infty$ in $\mathbf{H}_{-}^{1/2}(\Gamma)$. Further, for Γ_i, $\Gamma_j, j \in \mathcal{I}_i$, we set $\Gamma_{ij} := \Gamma_i \cup \Gamma_j \cup e_{ij}$. We have $\mathbf{t}_{ij}\cdot\mathbf{q}^k \in H^{1/2}(\Gamma_{ij})$. Hence, the uniqueness of the limit implies $\mathbf{t}_{ij}\cdot\mathbf{q} \in H^{1/2}(\Gamma_{ij})$ which gives the assertion.

Theorem 8.2 The tangential components trace mapping I

The tangential components trace mapping

$$\pi_{\mathbf{t}} : H^1(\Omega)^3 \to \mathbf{H}_{||}^{1/2}(\Gamma) \tag{8.66}$$

$$\mathbf{q} \longmapsto \mathbf{n}\wedge(\mathbf{q}\wedge\mathbf{n})|_\Gamma \tag{8.67}$$

is a surjective continuous linear mapping.

Proof. We have to show that for given $\boldsymbol{\varphi} \in \mathbf{H}_{||}^{1/2}(\Gamma)$ there exists $\mathbf{q} \in H^1(\Omega)^3$ such that $\mathbf{n}\wedge(\mathbf{q}\wedge\mathbf{n})|_\Gamma = \boldsymbol{\varphi}$.

By means of a partition of unity argument, we may restrict ourselves to the following three cases

$$\text{(i)} \quad \text{supp}\ \boldsymbol{\varphi} \subset \Gamma_i ,$$
$$\text{(ii)} \quad \text{supp}\ \boldsymbol{\varphi} \subset \Gamma_{ij} ,\ j \in \mathcal{I}_i ,$$

where $\Gamma_{ij} := \Gamma_i \cup \Gamma_j \cup e_{ij}$, and

$$\text{(iii)} \quad \text{supp}\ \boldsymbol{\varphi} \subset \hat{\Gamma}_i ,$$

where $\hat{\Gamma}_i$ is the union of the closed faces $\bar{\Gamma}_j$, $1 \le j \le N$, having $\mathbf{a}_i$ as a common vertex.

Reminding that $(\mathbf{t}_i, \mathbf{t}_{ij})$ is an orthonormal basis of the plane generated by Γ_i and $(\mathbf{t}_i, \mathbf{t}_{ij}, \mathbf{n}_i)$ is an orthonormal basis of $\mathbb{R}^3$, for $\mathbf{q} \in H^1(\Omega)^3$ and $\boldsymbol{\varphi} \in H_{||}^{1/2}(\Gamma)$ we have the local representations (observe that $\mathbf{n}_i\cdot\boldsymbol{\varphi} = 0$):

$$\mathbf{q}|_{\Gamma_i} = q_i\mathbf{t}_i + q_{ij}\mathbf{t}_{ij} + q_n\mathbf{n}_i , \tag{8.68}$$

$$\boldsymbol{\varphi}|_{\Gamma_i} = \varphi_i\mathbf{t}_i + \varphi_{ij}\mathbf{t}_{ij} . \tag{8.69}$$

Case (i): We may assume $\boldsymbol{\varphi} \in \mathbf{H}_{||}^{1/2}(\Gamma_i)$. In view of (8.68) and (8.69) we choose

$$q_i := \varphi_i \quad , \quad q_{ij} := \varphi_{ij} \quad , \quad q_n := 0$$

and thus get $\mathbf{q} \in H^1(\Omega)^3$ with $\mathbf{n} \wedge (\mathbf{q} \wedge \mathbf{n})|_{\Gamma_i} = \boldsymbol{\varphi}$.

Case (ii): We may assume $\boldsymbol{\varphi} \in \mathbf{H}_{||}^{1/2}(\Gamma_{ij})$. Again, with regard to (8.68) and (8.69) we choose

$$(8.70) \quad q_i := \varphi_i \mathbf{t}_i + \varphi_{ij}\mathbf{t}_{ij} + q_i\mathbf{n}_i \quad , \quad q_j := \varphi_j\mathbf{t}_j + \varphi_{ij}\mathbf{t}_{ij} + q_j\mathbf{n}_j$$

with q_i, q_j still to be determined.
Now, let α_{ij} be the angle between $\mathbf{t}_i$ and $\mathbf{t}_j$ and $c_{ij} := \cos\,\alpha_{ij}$, $s_{ij} := \sin\,\alpha_{ij}$ (cf. Figure 8.5).

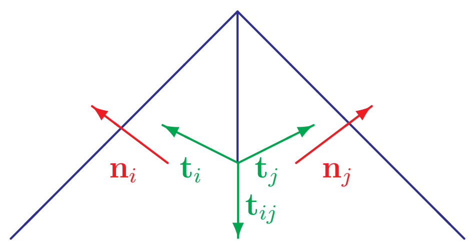

Fig. 8.5: Two adjacent faces Γ_i, Γ_j with common edge e_{ij}

We find that

$$(8.71) \qquad \mathbf{t}_j = c_{ij}\mathbf{t}_i - s_{ij}\mathbf{n}_i \ ,$$

$$(8.72) \qquad \mathbf{n}_j = c_{ij}\mathbf{n}_i + s_{ij}\mathbf{t}_i \ .$$

Using (8.71) and (8.71) in (8.70), it turns out that $\mathbf{q}|_{\Gamma_{ij}} \in H^{1/2}(\Gamma_{ij})$ if and only if

$$(8.73) \qquad \varphi_i =_{e_{ij}} c_{ij}\varphi_j + s_{ij}q_j \quad , \quad q_i =_{e_{ij}} - s_{ij}\varphi_j + c_{ij}q_j \ .$$

Without loss of generality we may assume that $s_{ij} \neq 0$. Hence, we may choose q_j according to the first equation and then q_i by means of the second one which gives the assertion.

Case (iii): In this case we may choose $\boldsymbol{\varphi} \in \mathbf{H}_{||}^{1/2}(\hat{\Gamma}_i)$. Further, without loss of generality we may assume that $\hat{\Gamma}_i = \hat{\Gamma}$ is a cone with a triangular transverse section consisting of three faces $\Gamma_i, 1 \leq i \leq 3$, their common edges e_{12}, e_{23}, e_{31} and the common vertex S, i.e.,

$$\hat{\Gamma} = (\Gamma_1 \cup \Gamma_2 \cup \Gamma_3) \cup (e_{12} \cup e_{23} \cup e_{31}) \cup \{S\} \ .$$

Denoting by α_1 the angle between $\mathbf{t}_1, \mathbf{t}_2$ and by α_2, α_3 the angles between $\mathbf{t}_2, \mathbf{t}_3$ and $\mathbf{t}_3, \mathbf{t}_1$ and setting $c_i := \cos\,\alpha_i$, $s_i := \sin\,\alpha_i$, $1 \leq i \leq 3$, as in case (ii) before, we get

$$\begin{aligned} \mathbf{t}_2 &= c_1\mathbf{t}_1 - s_1\mathbf{n}_1 \ , \\ \mathbf{t}_3 &= c_1\mathbf{t}_2 - s_1\mathbf{n}_2 \ , \\ \mathbf{t}_1 &= c_1\mathbf{t}_3 - s_1\mathbf{n}_3 \ . \end{aligned}$$

This leads to the six compatibility conditions:

$$(8.74) \qquad (C_1) \qquad \varphi_1 =_{e_{12}} c_1\varphi_2 + s_1 u_2 \ ,$$
$$(8.75) \qquad (C_2) \qquad u_1 =_{e_{12}} c_1 u_2 - s_1\varphi_2 \ ,$$
$$(8.76) \qquad (C_3) \qquad \varphi_2 =_{e_{23}} c_2\varphi_3 + s_2 u_3 \ ,$$
$$(8.77) \qquad (C_4) \qquad u_2 =_{e_{23}} c_2 u_3 - s_2\varphi_3 \ ,$$
$$(8.78) \qquad (C_5) \qquad \varphi_3 =_{e_{31}} c_3\varphi_1 + s_3 u_1 \ ,$$
$$(8.79) \qquad (C_6) \qquad u_3 =_{e_{31}} c_3 u_1 - s_3\varphi_1 \ .$$

We are able to decouple (8.74) - (8.79) by choosing $u_i^{(1)} \in H^{1/2}(\Gamma_i), 1 \leq i \leq 3$, such that the independent conditions C_1, C_3, and C_5 are satisfied. As a consequence, we have to compute $u_i^{(2)} \in H^{1/2}(\Gamma_i), 1 \leq i \leq 3$, such that

$$(8.80) \qquad (C_2)' \qquad u_1^{(2)} =_{e_{12}} c_1 u_2^{(1)} - s_1\varphi_2 \ ,$$
$$(8.81) \qquad (C_4)' \qquad u_2^{(2)} =_{e_{23}} c_2 u_3^{(1)} - s_2\varphi_3 \ ,$$
$$(8.82) \qquad (C_6)' \qquad u_3^{(2)} =_{e_{31}} c_3 u_1^{(1)} - s_3\varphi_1 \ .$$

This means that we have to find $u_i \in H^{1/2}(\Gamma_i), 1 \leq i \leq 3$, satisfying

$$\begin{aligned} u_1 &=_{e_{31}} u_1^{(1)} \quad , \quad & u_1 &=_{e_{12}} u_1^{(2)} \ , \\ u_2 &=_{e_{12}} u_2^{(1)} \quad , \quad & u_1 &=_{e_{23}} u_2^{(2)} \ , \\ u_3 &=_{e_{23}} u_3^{(1)} \quad , \quad & u_3 &=_{e_{31}} u_3^{(2)} \ . \end{aligned}$$

This can be done by means of a functions ξ_{ij} such that for all $\varphi \in H^{1/2}(\Gamma_i)$

$$(8.83) \qquad \xi_{ij}\varphi \in H^{1/2}(\Gamma_i) \quad , \quad \xi_{ij}|_{e_{ij}} = 1 \quad , \quad \xi_{ij}|_{e_{i\ell}} = 0 \ , \ \ell \neq j \ .$$

Indeed, if we set

$$\begin{aligned} u_1 &= \xi_{31}u_1^{(1)} + \xi_{12}u_1^{(2)} \ , \\ u_2 &= \xi_{12}u_2^{(1)} + \xi_{23}u_2^{(2)} \ , \\ u_3 &= \xi_{23}u_3^{(1)} + \xi_{31}u_3^{(2)} \ , \end{aligned}$$

then $u_i, 1 \leq i \leq 3$, satisfy (8.74 - (8.79). We have thus proven the existence of $\mathbf{q} \in H^1(\Omega)^3$ such that $\mathbf{n} \wedge (\mathbf{q} \wedge \mathbf{n})|_{\hat{\Gamma}} = \boldsymbol{\varphi}$. $\square$

Corollary 8.1 The tangential components trace mapping II

The tangential components trace mapping is a continuous, bijective linear mapping

$$\pi_{\mathbf{t}} \ : \ H^1(\Omega)^3/\mathrm{Ker}\ \pi_{\mathbf{t}} \ \rightarrow \ \mathbf{H}_{||}^{1/2}(\Gamma) \tag{8.84}$$

where $\mathrm{Ker}\ \pi_{\mathbf{t}} := \{\mathbf{q} \in H^1(\Omega)^3 \mid \mathbf{n} \wedge (\mathbf{q} \wedge \mathbf{n})|_\Gamma = 0\}$.

We now establish related mapping properties of the tangential trace mapping $\gamma_{\mathbf{t}}$. In view of Theorem 8.2 we introduce the space

$$\begin{aligned}\mathbf{H}_\perp^{1/2}(\Gamma) \ &:= \\ &:= \ \{ \ \mathbf{q} \in \mathbf{H}_-^{1/2}(\Gamma) \mid \mathbf{t}_i \cdot \mathbf{q}_i \ =_{e_{ij}} \ \mathbf{t}_j \cdot \mathbf{q}_j \ , \ 1 \le i \le N \ , \ j \in \mathcal{I}_i \ \} \ .\end{aligned} \tag{8.85}$$

Lemma 8.2 The space $\mathbf{H}_\perp^{1/2}(\Gamma)$

The space $\mathbf{H}_\perp^{1/2}(\Gamma)$ is a Hilbert space with respect to the norm

$$\|\mathbf{q}\|_{\perp,1/2,\Gamma} \ := \\ \sum_{i=1}^N \|\mathbf{q}_i\|_{1/2,\Gamma_i}^2 + \sum_{i=1}^N \sum_{j \in \mathcal{I}_i} \int_{\Gamma_i} \int_{\Gamma_j} \frac{|\mathbf{t}_i \cdot \mathbf{q}_i(\mathbf{x}) - \mathbf{t}_j \cdot \mathbf{q}_j(\mathbf{y})|^2}{|\mathbf{x} - \mathbf{y}|^3} \ d\sigma(\mathbf{x}) \ d\sigma(\mathbf{y}).$$

Theorem 8.3 The tangential trace mapping I

The tangential trace mapping $\gamma_{\mathbf{t}}$ is a continuous, surjective linear mapping

$$\gamma_{\mathbf{t}} \ : \ H^1(\Omega)^3 \ \rightarrow \ \mathbf{H}_\perp^{1/2}(\Gamma) \tag{8.86}$$

Corollary 8.2 The tangential trace mapping II

The tangential trace mapping $\gamma_{\mathbf{t}}$ is a continuous, bijective linear mapping

$$\gamma_{\mathbf{t}} \ : \ H^1(\Omega)^3/\mathrm{Ker}\ \gamma_{\mathbf{t}} \ \rightarrow \ \mathbf{H}_\perp^{1/2}(\Gamma) \tag{8.87}$$

where $\mathrm{Ker}\ \gamma_{\mathbf{t}} := \{\mathbf{q} \in H^1(\Omega)^3 \mid \mathbf{q} \wedge \mathbf{n}|_\Gamma = 0\}$.

The proofs of Lemma 8.2, Theorem 8.3, and Corollary 8.2 are left as easy exercises.

In the sequel we will refer to $\mathbf{H}_{||}^{-1/2}(\Gamma)$ and $\mathbf{H}_\perp^{-1/2}(\Gamma)$ as the dual spaces of $\mathbf{H}_{||}^{1/2}(\Gamma)$ and $\mathbf{H}_\perp^{1/2}(\Gamma)$ with $\mathbf{L}_{\mathbf{t}}^2(\Gamma)$ as the pivot space.

8.2.3 Tangential differential operators

For a smooth function $u \in \mathcal{D}(\bar{\Omega})$ the tangential gradient operator $\nabla_\Gamma = \mathbf{grad}|_\Gamma$ is defined as the tangential components trace of the gradient operator ∇

$$\nabla_\Gamma u \;:=\; \pi_{\mathbf{t}}(\nabla u) \tag{8.88}$$

where (8.88) has to be understood facewise

$$\nabla_\Gamma u|_{\Gamma_i} \;:=\; \nabla_{\Gamma_i} u \;=\; \pi_{\mathbf{t},i}(\nabla u) \;=\; \mathbf{n}_i \wedge (\nabla u \wedge \mathbf{n}_i) \;,\; 1 \le i \le N \;.$$

Since $\mathcal{D}(\bar{\Omega})$ is dense in $H^2(\Omega)$, we easily get

Theorem 8.4 The tangential gradient operator

The tangential gradient operator is a continuous linear mapping

$$\nabla_\Gamma \;:\; H^{3/2}(\Gamma) \;\to\; \mathbf{H}_{||}^{1/2}(\Gamma) \;. \tag{8.89}$$

Proof. Since $\nabla_{\Gamma_i} : H^2(\Omega) \to H^{1/2}(\Gamma_i)^3$ and $\mathbf{n}_i \cdot \pi_{\mathbf{t},i}(\nabla u)|_{\Gamma_i} = 0$, we have $\nabla_\Gamma : H^2(\Omega) \to \mathbf{H}_-^{1/2}(\Gamma)$. In view of $u|_\Gamma \in H^{3/2}(\Gamma)$ for $u \in H^2(\Omega)$, the assertion follows from the mapping properties of the tangential components trace mapping $\pi_{\mathbf{t}}$ (cf. Theorem 8.3).

Definition 8.3 The tangential divergence operator

The tangential divergence operator

$$\mathrm{div}|_\gamma \;:\; \mathbf{H}_{||}^{-1/2}(\Gamma) \;\to\; H^{-3/2}(\Gamma) \tag{8.90}$$

is defined as the adjoint operator of $-\nabla_\Gamma$

$$< \mathrm{div}|_\Gamma \mathbf{q}, u >_{3/2,\Gamma} = - < \mathbf{q}, \nabla_\Gamma u >_{||,1/2,\Gamma} \;,\; u \in H^{3/2}(\Gamma) \;,\; \mathbf{q} \in \mathbf{H}_{||}^{-1/2}(\Gamma) \;.$$

Finally, for $u \in \mathcal{D}(\Omega)$ we define the tangential curl operator $\mathbf{curl}|_\Gamma$ as the tangential trace of the gradient operator

$$\mathbf{curl}|_\Gamma u \;=\; \gamma_{\mathbf{t}}(\nabla u) \tag{8.91}$$

where again (8.91) must be understood facewise

$$\mathbf{curl}_\Gamma u|_{\Gamma_i} \;=\; \mathbf{curl}|_{\Gamma_i} u \;=\; \gamma_{\mathbf{t},i}(\nabla u) \;=\; \nabla u \wedge \mathbf{n}_i \;,\; 1 \le i \le N \;.$$

Theorem 8.5 The vectorial tangential curl operator

The vectorial tangential curl operator is a linear continuous mapping

$$\mathbf{curl}_\Gamma \;:\; H^{3/2}(\Gamma) \;\to\; \mathbf{H}_\perp^{1/2}(\Gamma) \;. \tag{8.92}$$

The proof of this result follows the same lines as the proof of Theorem 1.17.

Definition 8.4 The scalar tangential curl operator
The scalar tangential curl operator

(8.93) $$\mathrm{curl}_\Gamma \ : \ \mathbf{H}_\perp^{-1/2}(\Gamma) \ \rightarrow \ H^{-3/2}(\Gamma)$$

is defined as the adjoint of the vectorial tangential curl operator $\mathbf{curl}|_\gamma$

$$< \mathrm{curl}|_\Gamma \mathbf{q}, u >_{3/2,\Gamma} = < \mathbf{q}, \mathbf{curl}|_\Gamma u >_{\perp,1/2,\Gamma} \ , \ u \in H^{3/2}(\Gamma) \ , \ \mathbf{q} \in \mathbf{H}_\perp^{-1/2}(\Gamma).$$

8.2.4 Trace mappings of $\mathbf{H}(\mathbf{curl};\Omega)$

In this section we consider the tangential trace mapping $\gamma_\mathbf{t}$ and the tangential components trace mapping $\pi_\mathbf{t}$ on $\mathbf{H}(\mathbf{curl};\Omega)$ and characterize its range spaces. To this end we introduce the spaces

(8.94) $$\mathbf{H}_{||}^{-1/2}(\mathrm{div}|_\Gamma,\Gamma) \ := \ \{ \ \boldsymbol{\lambda} \in \mathbf{H}_{||}^{-1/2}(\Gamma) \mid \mathrm{div}|_\Gamma \boldsymbol{\lambda} \in H^{-1/2}(\Gamma) \ \} \ ,$$

(8.95) $$\mathbf{H}_{\perp}^{-1/2}(\mathrm{curl}|_\Gamma,\Gamma) \ := \ \{ \ \boldsymbol{\lambda} \in \mathbf{H}_{\perp}^{-1/2}(\Gamma) \mid \mathrm{curl}|_\Gamma \boldsymbol{\lambda} \in H^{-1/2}(\Gamma) \ \} \ .$$

Theorem 8.6 The tangential trace mapping III
The tangential trace mapping is a continuous linear mapping

(8.96) $$\gamma_\mathbf{t} \ : \ \mathbf{H}(\mathbf{curl},\Omega) \ \rightarrow \ \mathbf{H}_{||}^{-1/2}(\mathrm{div}|_\Gamma,\Gamma) \ .$$

Proof. For $\mathbf{j} \in \mathcal{D}(\bar{\Omega})^3$ and $\boldsymbol{\lambda} := \gamma_\mathbf{t}(\mathbf{j})$ the Stokes theorem gives

(8.97) $$\int_\Omega [\, \mathbf{curl}\, \mathbf{q}\cdot\mathbf{j} - \mathbf{q}\cdot\mathbf{curl}\, \mathbf{j} \,] \, d\mathbf{x} \ = \ \int_\Gamma \lambda\cdot\pi_\mathbf{t}(\mathbf{q}) \, d\sigma \quad , \quad \mathbf{q} \in H^1(\Omega)^3 \ .$$

Since $\pi_\mathbf{t} : H^1(\Omega)^3/\mathrm{Ker}\ \pi\mathbf{t} \rightarrow \mathbf{H}_{||}^{1/2}(\Gamma)$ is continuous, linear, and bijective, we have

$$\|\boldsymbol{\lambda}\|_{||,-1/2,\Gamma} \ = \ \sup_{\boldsymbol{\mu}\in\mathbf{H}_{||}^{1/2}(\Gamma)} \frac{< \boldsymbol{\lambda}, \mathbf{u} >_{||,1/2,\Gamma}}{\|\boldsymbol{\mu}\|_{||,1/2,\Gamma}} \ \leq$$
$$\leq \ C \sup_{\mathbf{q}\in H^1(\Omega)^3/Ker\pi_\mathbf{t}} \frac{< \boldsymbol{\lambda}, \pi_\mathbf{t}(\mathbf{q}) >_{||,1/2,\Gamma}}{\|\mathbf{q}\|_{1,\Omega}} \ .$$

Taking (8.97) into account, it follows that

(8.98) $$\|\boldsymbol{\lambda}\|_{||,-1/2,\Gamma} \ \leq \ C \ \|\mathbf{j}\|_{curl,\Omega}$$

which proves $\boldsymbol{\lambda} \in \mathbf{H}_{||}^{-1/2}(\Gamma)$.
Next, we have to show that $\mathrm{div}|_\Gamma(\mathbf{j} \wedge \mathbf{n}) \in H^{-1/2}(\Gamma)$. Applying Stokes'

theorem once more by choosing $\mathbf{q} = \nabla\varphi$, $\varphi \in H^2(\Omega)$ and taking (8.90) into account, we obtain

$$(8.99) \qquad \int_\Omega \mathbf{curl}\ \mathbf{j} \cdot \nabla\varphi\ d\mathbf{x} \ = \ - \int_\Gamma (\mathbf{j} \wedge \mathbf{n}) \cdot \pi_{\mathbf{t}}(\nabla\varphi)\ d\sigma \ = \ = \ - < \mathbf{j} \wedge \mathbf{n}, \nabla_\Gamma \varphi >_{||,1/2,\Gamma} \ = \ < \mathrm{div}_\Gamma(\mathbf{j} \wedge \mathbf{n}), \varphi >_{3/2,\Gamma} \ .$$

In particular, $\varphi|_\Gamma \in H^{1/2}(\Gamma)$ so that there exists $v \in H^1(\Omega)$ with $v|_\Gamma = \varphi|_\Gamma$ and $\|v\|_{1,\Omega} \le C\|\varphi\|_{1/2,\Gamma}$. If we set $v_0 := v - \varphi$, then $v_0 \in H^1_0(\Omega)$ and (8.99) results in

$$< \mathrm{div}_\Gamma(\mathbf{j} \wedge \mathbf{n}), \varphi >_{3/2,\Gamma} \ = \ \int_\Omega \mathbf{curl}\ \mathbf{j} \cdot \nabla(\varphi + v_0)\ d\mathbf{x} \ \le \ \le \ \|\mathbf{j}\|_{curl,\Omega} \|v\|_{1,\Omega} \ \le \ C\ \|\mathbf{j}\|_{curl,\Omega} \|\varphi\|_{1/2,\Gamma} \ .$$

Since $H^2(\Omega)|_\Gamma$ is dense in $H^{1/2}(\Gamma) = H^1(\Omega)|_\Gamma$, the previous inequality proves that the functional $\mathrm{div}_\Gamma(\mathbf{j} \wedge \mathbf{n})$ can be extended to a continuous linear functional on $H^{1/2}(\Gamma)$ and that

$$(8.100) \qquad \mathrm{div}_\Gamma(\mathbf{j} \wedge \mathbf{n}) \in H^{-1/2}(\Gamma) \ , \qquad \|\mathrm{div}_\Gamma(\mathbf{j} \wedge \mathbf{n})\|_{-1/2,\Gamma} \ \le \ C\ \|\mathbf{j}\|_{curl,\Omega} \ , \ \mathbf{j} \in \mathcal{D}(\bar{\Omega})^3 \ .$$

Recalling that $\mathcal{D}(\bar{\Omega})^3$ is dense in $\mathbf{H}(\mathbf{curl};\Omega)$, it follows that (8.98) and (8.100) also hold true for $\mathbf{j} \in H(\mathbf{curl};\Omega)$, and we conclude. □

Corollary 8.3 Generalization of Stokes' theorem I

Stokes' theorem can be generalized as follows: For $\mathbf{j} \in \mathbf{H}(\mathbf{curl},\Omega)$, $\mathbf{q} \in H^1(\Omega)^3$ there holds

$$(8.101) \qquad \int_\Omega [\ \mathbf{curl}\ \mathbf{q} \cdot \mathbf{j} \ - \ \mathbf{q} \cdot \mathbf{curl j}\ ]\ d\mathbf{x} \ = < \gamma_{\mathbf{t}}(\mathbf{q}), \pi_{\mathbf{t}}(\mathbf{j}) >_{||,1/2,\Gamma} \ .$$

In much the same way, the following result can be established:

Theorem 8.7 The tangential components trace mapping III

The tangential components trace mapping is a continuous linear mapping

$$(8.102) \qquad \pi_{\mathbf{t}} \ : \ \mathbf{H}(\mathbf{curl};\Omega) \ \rightarrow \ \mathbf{H}_\perp^{-1/2}(\mathrm{curl}|_\Gamma, \Gamma) \ .$$

Proof. For $\mathbf{j} \in \mathcal{D}(\bar{\Omega})^3$ and $\boldsymbol{\lambda} := \pi_{\mathbf{t}}(\mathbf{j})$ Stokes' theorem gives

$$\int_\Omega [\ \mathbf{curl}\ \mathbf{j} \cdot \mathbf{q} \ - \ \mathbf{j} \cdot \mathbf{curl}\ \mathbf{q}\ ]\ d\mathbf{x} \ = \ \int_\Gamma \gamma_{\mathbf{t}}(\mathbf{q}) \cdot \boldsymbol{\lambda}\ d\sigma \quad , \quad \mathbf{q} \in H^1(\Omega)^3 \ .$$

Using that $\gamma_{\mathbf{t}} : H^1(\Omega)^3/\mathrm{Ker}\gamma_{\mathbf{t}} \to \mathbf{H}_\perp^{-1/2}(\Gamma)$ is continuous, linear, and bijective, we find $\boldsymbol{\lambda} \in \mathbf{H}_\perp^{-1/2}(\Gamma)$.
Moreover, for $\varphi \in H^2(\Omega)$ and $\mathbf{q} := \nabla\varphi$

$$
\begin{aligned}
(8.103) \int_\Omega \mathbf{curl}\, \mathbf{j} \cdot \nabla\varphi \, d\mathbf{x} \; &= \; \int_\Gamma (\mathbf{n} \wedge (\mathbf{j} \wedge \mathbf{n})) \cdot \nabla\varphi \, d\sigma \; = \\
&= \; < \pi_{\mathbf{t}}(\mathbf{j}), \mathrm{curl}|_\Gamma \, \varphi >_{\perp,1/2,\Gamma} \; = \\
&= \; < \mathrm{curl}|_\Gamma \, (\mathbf{n} \wedge (\mathbf{j} \wedge \mathbf{n})), \varphi >_{3/2,\Gamma} \quad .
\end{aligned}
$$

In the same way as in the proof of the previous theorem we can show that $\mathrm{curl}|_\Gamma \, (\mathbf{n} \wedge (\mathbf{j} \wedge \mathbf{n})) \in H^{-1/2}(\Gamma)$. We conclude by the standard density argument. □

Corollary 8.4 Generalization of Stokes' theorem II

Stokes' theorem can be generalized as follows: For $\mathbf{j} \in \mathbf{H}(\mathbf{curl}, \Omega)$, $\mathbf{q} \in H^1(\Omega)^3$ there holds

$$
(8.104) \qquad \int_\Omega [\, \mathbf{curl}\, \mathbf{j} \cdot \mathbf{q} \; - \; \mathbf{j} \cdot \mathbf{curl} \mathbf{q} \,] \, d\mathbf{x} \; = < \gamma_{\mathbf{t}}(\mathbf{j}), \pi_{\mathbf{t}}(\mathbf{q}) >_{\perp,1/2,\Gamma} \quad .
$$

The previous results imply that the tangential divergence of the tangential trace and the scalar tangential curl of the tangential components trace coincide.

Corollary 8.5 Properties of the tangential operators

For $\mathbf{j} \in \mathbf{H}(\mathbf{curl}; \Omega)$ there holds

$$
(8.105) \qquad \mathrm{div}|_\Gamma \, (\mathbf{j} \wedge \mathbf{n}) \; = \; \mathrm{curl}|_\Gamma \, (\mathbf{n} \wedge (\mathbf{j} \wedge \mathbf{n})) \; = \; \mathbf{n} \cdot \mathbf{curl}\, \mathbf{j} \; .
$$

Proof. Using (8.99), for $\mathbf{j} \in \mathcal{D}(\bar{\Omega})^3$ and $\varphi \in H^2(\Omega)$ we have

$$
- \int_\Gamma (\mathbf{j} \wedge \mathbf{n}) \cdot \nabla_\Gamma \varphi \, d\sigma \; = \; \int_\Omega \mathbf{curl}\, \mathbf{j} \cdot \nabla\varphi \, d\mathbf{x} \; = \; \int_\Gamma (\mathbf{curl}\, \mathbf{j} \cdot \mathbf{n}) \varphi \, d\sigma \; .
$$

Again, (8.99) and the density of $H^2(\Omega)|_\Gamma$ in $H^{1/2}(\Gamma)$ give

$$
< \mathrm{div}|_\Gamma \, (\mathbf{j} \wedge \mathbf{n}), \varphi >_{1/2,\Gamma} \, = < \mathbf{curl}\, \mathbf{j} \cdot \mathbf{n}), \varphi >_{1/2,\Gamma} \; ,
$$

and hence, the density of $\mathcal{D}(\bar{\Omega})^3$ in $\mathbf{H}(\mathbf{curl}; \Omega)$ implies

$$
\mathrm{div}|_\Gamma \, (\mathbf{j} \wedge \mathbf{n}) \; = \; \mathbf{curl}\, \mathbf{j} \cdot \mathbf{n} \; .
$$

On the other hand, using (8.103), for $\mathbf{j} \in \mathcal{D}(\bar{\Omega})^3$ and $\varphi \in H^2(\Omega)$

$$
\int_\Gamma (\mathbf{n} \wedge (\mathbf{j} \wedge \mathbf{n})) \cdot \nabla_\Gamma \varphi \, d\sigma \; = \; \int_\Omega \mathbf{curl}\, \mathbf{j} \cdot \nabla\varphi \, d\mathbf{x} \; = \; \int_\Gamma (\mathbf{curl}\, \mathbf{j} \cdot \mathbf{n}) \varphi \, d\sigma \; .
$$

Applying the right-hand side in (8.103) and taking again advantage of the density of $H^2(\Omega)|_\Gamma$ in $H^{1/2}(\Gamma)$ and $\varphi \in H^2(\Omega)$

$$< \mathrm{curl}|_\Gamma\ (\mathbf{n} \wedge (\mathbf{j} \wedge \mathbf{n})), \varphi >_{1/2,\Gamma} = < \mathbf{curl}\ \mathbf{j} \cdot \mathbf{n}, \varphi >_{1/2,\Gamma}$$

whence, by density of $\mathcal{D}(\bar{\Omega})^3$ in $\mathbf{H}(\mathbf{curl};\Omega)$

$$\mathrm{curl}|_\Gamma\ (\mathbf{n} \wedge (\mathbf{j} \wedge \mathbf{n}))\ =\ \mathbf{curl}\ \mathbf{j} \cdot \mathbf{n}\ .$$

8.3 Edge elements and edge element spaces

8.3.1 Conforming elements for $\mathbf{H}(\mathbf{curl};\Omega)$

Let $\mathcal{T}_h$ be a triangulation of Ω. For $D \subset \bar{\Omega}$, we refer to $\mathcal{E}_h(D)$ and $\mathcal{F}_h(D)$ as the sets of edges and faces of $\mathcal{T}_h$ in D.
We consider

(8.106) $$P_K\ :=\ \{\ \mathbf{q} = (q_1, ..., q_d)^T \mid q_i : K \to \mathbb{R}\ ,\ 1 \le i \le d\ \}\ ,\ K \in \mathcal{T}_h\ ,$$

(8.107) $$V_h(\Omega)\ :=\ \{\ \mathbf{q}_h : \bar{\Omega} \to \mathbb{R} \mid \mathbf{q}_h|_K \in P_K\ ,\ K \in \mathcal{T}_h\ \}\ .$$

The following result gives sufficient conditions for $V_h(\Omega) \subset \mathbf{H}(\mathbf{curl};\Omega)$.

Theorem 8.8 Sufficient conditions for conformity

Let $\mathcal{T}_h$ be a triangulation of Ω and let P_K , $K \in \mathcal{T}_h$, and $V_h(\Omega)$ be given by (8.106) and (8.107), respectively. Assume that

(8.108) $$P_K \subset \mathbf{H}(\mathbf{curl};K)\ ,\ K \in \mathcal{T}_h\ ,$$

(8.109) $$[\mathbf{q}_h \wedge \mathbf{n}|_F]_J\ =\ 0 \quad \text{for all } F = K_i \cap K_j \in \mathcal{F}_h(\Omega)\ ,\ \mathbf{q}_h \in V_h(\Omega)\ ,$$

where $\mathbf{n}$ is the unit normal on F pointing towards K_i and $[\mathbf{q}_h \wedge \mathbf{n}|_F]_J$ denotes the jump of $\mathbf{q}_h \wedge \mathbf{n}$ across F, i.e.,

(8.110) $$[\mathbf{q}_h \wedge \mathbf{n}|_F]_J := (\mathbf{q}_h \wedge \mathbf{n}|_{F\cap Ki} - \mathbf{q}_h \wedge \mathbf{n})|_{F\cap Kj}\ .$$

Then $V_h(\Omega) \subset \mathbf{H}(\mathbf{curl};\Omega)$.

Proof. Given $\mathbf{q}_h \in V_h(\Omega)$, we have to show that $\mathbf{curl}\mathbf{q}_h$ is well defined and $\mathbf{curl}\mathbf{q}_h \in L^2(\Omega)^3$. In other words, we have to find $\mathbf{z}_h \in L^2(\Omega)^3$ such that

$$\int_\Omega \mathbf{q}_h \cdot \mathbf{curl}\ \boldsymbol{\varphi}\ d\mathbf{x}\ =\ \int_\Omega \mathbf{z}_h \cdot \boldsymbol{\varphi}\ d\mathbf{x} \quad , \quad \boldsymbol{\varphi} \in \mathcal{D}(\Omega)^3\ .$$

In view of (8.108), Stokes's formula can be applied elementwise:

$$\int_\Omega \mathbf{q}_h \cdot \mathbf{curl}\, \boldsymbol{\varphi}\, d\mathbf{x} = \sum_{K \in \mathcal{T}_h} \int_K \mathbf{q}_h \cdot \mathbf{curl}\, \boldsymbol{\varphi}\, d\mathbf{x} =$$

$$= \sum_{K \in \mathcal{T}_h} \int_K \mathbf{curl} \mathbf{q}_h \cdot \boldsymbol{\varphi}\, d\mathbf{x} + \sum_{K \in \mathcal{T}_h} \int_{\partial K} (\mathbf{q}_h \wedge \mathbf{n})|_{\partial K} \cdot (\mathbf{n} \wedge (\boldsymbol{\varphi} \wedge \mathbf{n}))\, d\sigma =$$

$$= \sum_{K \in \mathcal{T}_h} \int_K \mathbf{curl} \mathbf{q}_h \cdot \boldsymbol{\varphi}\, d\mathbf{x} + \sum_{F \in \mathcal{F}_h(\Omega)} \int_F [\mathbf{q}_h \wedge \mathbf{n}|_F]_J \cdot (\mathbf{n} \wedge (\boldsymbol{\varphi} \wedge \mathbf{n}))\, d\sigma\ .$$

Taking advantage of (8.109), the assertion follows for $\mathbf{z}_h$ with $\mathbf{z}_h|_K := \mathbf{curl} \mathbf{q}_h|_K, K \in \mathcal{T}_h$.

8.3.2 The edge elements $\mathbf{Nd}_k(K)$ of Nédélec's first family for simplicial triangulations

Let us consider a simplicial triangulation $\mathcal{T}_h$ of Ω. For $k \in \mathbb{N}$, we refer to $P_k(K)$ resp. $\tilde{P}_k(K)$ as the set of polynomials of degree k on K resp. the set of homogeneous polynomials of degree k on K, i.e.,

$$\tilde{P}_k(K) \;:=\; \{\, p(\mathbf{x}) \;=\; \sum_{|\alpha|=k} a_\alpha \mathbf{x}^\alpha \;,\; \mathbf{x} \in K \,\} \;,$$
$$\dim\, \tilde{P}_k(K) \;=\; \binom{k+d-1}{k} \;.$$

We define $S_k(K)$ as the space

(8.111) $$S_k(K) \;:=\; \{\, \mathbf{q} \in \tilde{P}_k(K)^d \mid \mathbf{x}\cdot\mathbf{q} \equiv 0 \;,\; \mathbf{x} \in K \,\} \;,$$

(8.112) $$\dim\, S_k(K) \;=\; \begin{cases} k & , \quad d = 2 \\ k(k+2) & , \quad d = 3 \end{cases} \;.$$

Definition 8.5 Edge elements

Let K be a d-simplex. The edge element $\mathbf{Nd}_k(K)$, $k \in \mathbb{N}$, of Nédélec's first family is given by

(8.113) $$\mathbf{Nd}_k(K) \;=\; P_{k-1}(K)^d \;+\; S_k(K) \;.$$

For $\mathbf{q} \in \mathbf{Nd}_k(K)$, the degrees of freedom Σ_K are given by

(i) d = 2

(8.114) $$\int_E \mathbf{q}\cdot\mathbf{t}_E \; p_{k-1} \; ds \quad , \quad p_{k-1} \in P_{k-1}(E) \;,\; E \in \mathcal{E}_h(K) \;,$$

(8.115) $$\int_K \mathbf{q}\cdot\mathbf{p}_{k-2} \; d\sigma \quad , \quad \mathbf{p}_{k-2} \in P_{k-2}(K)^2 \;,\; K \in \mathcal{T}_h(K) \;.$$

(ii) d = 3

(8.116) $$\int_E \mathbf{q}\cdot\mathbf{t}_E \; p_{k-1} \; ds \quad , \quad p_{k-1} \in P_{k-1}(E) \;,\; E \in \mathcal{E}_h(K) \;,$$

(8.117) $$\int_F (\mathbf{q}\wedge\mathbf{n})\cdot\mathbf{p}_{k-2} \; d\sigma \quad , \quad \mathbf{p}_{k-2} \in P_{k-2}(F)^2 \;,\; F \in \mathcal{F}_h(K) \;,$$

(8.118) $$\int_K \mathbf{q}\cdot\mathbf{p}_{k-3} \; d\mathbf{x} \quad , \quad \mathbf{p}_{k-3} \in P_{k-3}(K)^3 \;.$$

where $\mathbf{t}_E$ is a unit vector parallel to $E \in \mathcal{E}_h(K)$.
We have

$$\dim\, Nd_k(K) \;=\; \begin{cases} k\,(k+2) & , \quad d=2 \\ \frac{1}{2}\,k\,(k+2)\,(k+3) & , \quad d=3 \end{cases} \quad . \tag{8.119}$$

Examples of edge element spaces

(i) k = 1 , d = 2

Let $\mathbf{p} = (p_1, p_2) \in S_1(K)$, i.e.,

$$\mathbf{p} \;=\; \begin{pmatrix} a_1x_1 + b_1x_2 \\ a_2x_1 + b_2x_2 \end{pmatrix} \quad , \quad \mathbf{x}\cdot\mathbf{p} \;=\; 0\;,\; \mathbf{x} \in K\;.$$

The condition $\mathbf{x}\cdot\mathbf{p} = 0, \mathbf{x} \in K$, leads to

$$a_1 \;=\; 0 \quad , \quad b_2 \;=\; 0 \quad , \quad b_1 \;=\; -\,a_2 \quad ,$$

and hence

$$S_1(K) \;=\; \text{span}\;\{\begin{pmatrix} -\,x_2 \\ x_1 \end{pmatrix}\} \quad .$$

It follows that

$$\mathbf{Nd}_1(K) \;=\; \{\;\mathbf{q} \;=\; \mathbf{a} \;+\; b\;\begin{pmatrix} -\,x_2 \\ x_1 \end{pmatrix}\;|\;\mathbf{a} \in \mathbb{R}^2\;,\; b \in \mathbb{R}\;\} \quad .$$

(ii) k = 2 , d = 2

Any $\mathbf{p} = (p_1, p_2) \in S_2(K)$ satisfies

$$\mathbf{p} \;=\; \begin{pmatrix} a_1x_1^2 + b_1x_1x_2 + c_1x_2^2 \\ a_2x_1^2 + b_2x_1x_2 + c_2x_2^2 \end{pmatrix} \quad , \quad \mathbf{x}\cdot\mathbf{p} \;=\; 0\;,\; \mathbf{x} \in K\;.$$

Using the same reasoning as in (i), we obtain

$$S_1(K) \;=\; \text{span}\;\{\begin{pmatrix} x_2^2 \\ -x_1x_2 \end{pmatrix} \;,\; \begin{pmatrix} -x_1x_2 \\ x_1^2 \end{pmatrix}\} \quad .$$

(iii) k = 1 , d = 3

$\mathbf{p} = (p_1, p_2, p_3) \in \tilde{P}_1(K)$ has the representation

$$\begin{pmatrix} a_1x_1 + b_1x_2 + c_1x_3 \\ a_2x_1 + b_2x_2 + c_2x_3 \\ a_3x_1 + b_3x_2 + c_3x_3 \end{pmatrix} \quad .$$

The requirement $\mathbf{x}\cdot\mathbf{p} = \sum_{i=1}^3 x_ip_i = 0$ leads to

$$\begin{aligned} &a_1x_1^2 + b_2x_2^2 + c_3x_3^2 + (b_1 + a_2)x_1x_2 + \\ &+(c_1 + a_3)x_1x_3 + (c_2 + b_3)x_2x_3 = 0\;,\; \mathbf{x} \in K\;. \end{aligned}$$

We conclude

$$a_1 = b_2 = c_3 = 0 \ ,$$
$$b_1 + a_2 = c_1 + a_3 = c_2 + b_3 = 0 \ ,$$

whence

$$\mathbf{p} \ = \ \mathbf{b} \wedge \mathbf{x} \quad , \quad \mathbf{b} \in \mathbb{R}^3 \quad .$$

Consequently, we get

$$\mathbf{Nd}_1(K) \ = \ \{ \ \mathbf{q} \ = \ \mathbf{a} \ + \ \mathbf{b} \wedge \mathbf{x} \mid \mathbf{a}, \mathbf{b} \in \mathbb{R}^3 \ \} \quad .$$

(iv) k = 2 , d = 3

It is sufficient to elaborate on $S_2(K)$. According to (8.112), we have

$$\dim \ S_2(K) \ = \ 8 \quad .$$

In much the same way as in (iii), we find that $S_2(K)$ is spanned by the following basis:

$$\begin{pmatrix} x_2^2 \\ -x_1x_2 \\ 0 \end{pmatrix} , \begin{pmatrix} x_2^2 \\ -x_2x_3 \\ x_2^2 \end{pmatrix} , \begin{pmatrix} -x_1x_2 \\ x_1^2 \\ 0 \end{pmatrix} , \begin{pmatrix} x_1x_3 \\ 0 \\ x_1^2 \end{pmatrix} ,$$
$$\begin{pmatrix} x_3^2 \\ 0 \\ -x_1x_3 \end{pmatrix} , \begin{pmatrix} 0 \\ -x_3^2 \\ -x_2x_3 \end{pmatrix} , \begin{pmatrix} x_2x_3 \\ -x_1x_3 \\ 0 \end{pmatrix} , \begin{pmatrix} 0 \\ x_1x_3 \\ -x_1x_2 \end{pmatrix} .$$

In the general case, we will first verify that (8.109) is satisfied, i.e., the edge elements $\mathbf{Nd}_k(K)$ are conforming. For this purpose it is sufficient to show:

Theorem 8.9 Conformity of the edge elements

Let $K \in \mathcal{T}_h$ and $F \in \mathcal{F}_h(K)$ and suppose that

$$(8.120) \quad \int_E \mathbf{q} \cdot \mathbf{t}_E \ p_{k-1} \ ds \ = \ 0 \quad , \quad p_{k-1} \in P_{k-1}(E) \ , \ E \in \mathcal{E}_h(F) \ ,$$

$$(8.121) \quad \int_F (\mathbf{q} \wedge \mathbf{n}) \cdot \mathbf{p}_{k-2} \ d\sigma \ = \ 0 \quad , \quad \mathbf{p}_{k-2} \in P_{k-2}(F)^2 \ .$$

Then, there holds

$$(8.122) \qquad \mathbf{q} \wedge \mathbf{n} \ = \ 0 \quad \text{on } F \ .$$

Proof. Since $\mathbf{q} \cdot \mathbf{t}_E \in P_{k-1}(E), E \in \mathcal{E}_h(F)$, (8.120) implies

$$(8.123) \qquad \mathbf{q} \cdot \mathbf{t}_E \ = \ 0 \quad \text{on } E \in \mathcal{E}_h(F) \ .$$

Now, Green's theorem implies

$$\text{(8.124)} \qquad \int_F (\mathbf{grad}_F p \cdot (\mathbf{q} \wedge \mathbf{n}) + p \ \mathrm{div}_F(\mathbf{q} \wedge \mathbf{n}) \ d\sigma \ = $$

$$= \int_{\partial F} p \ \mathbf{n}_{\partial F} \cdot (\mathbf{q} \wedge \mathbf{n}) \ ds \ = \ \int_{\partial F} p \ \mathbf{q} \cdot \mathbf{t} \ ds \ , \ p \in P_{k-1}(F) \ .$$

Since $\mathbf{grad}_F p \in P_{k-2}(F)^2$, (8.121) and (8.123) imply

$$\text{(8.125)} \qquad \mathrm{div}_F(\mathbf{q} \wedge \mathbf{n}) \ = \ 0 \quad \text{on } F \ ,$$

whence

$$\text{(8.126)} \qquad \mathbf{q} \wedge \mathbf{n} \ = \ \mathbf{curl}_F \varphi \quad , \quad \varphi \in P_k(F) \ .$$

Moreover, (8.123) tells us

$$0 \ = \ \mathbf{t}_E \cdot \mathbf{q}|_E \ = \ \mathbf{n}_E \cdot (\mathbf{q} \wedge \mathbf{n})|_E \ = \ \mathbf{n}_E \cdot (\mathbf{curl}_F \ \varphi)|_E \quad , \quad E \in \mathcal{E}_h(F) \ ,$$

and hence

$$(\mathbf{curl}_F \ \varphi)|_E \ = \ 0 \ , \ E \in \mathcal{E}_h(F) \quad \Longrightarrow \quad \varphi|_{\partial F} \ = \ const. \ .$$

Since φ is uniquely determined up to a constant, we may choose

$$\varphi|_{\partial F} \ = \ 0 \ .$$

Denoting by $\lambda_i^F, 1 \leq i \leq 3,$ the barycentric coordinates of the triangle F, it follows that

$$\text{(8.127)} \qquad \varphi \ = \ \lambda_1^F \lambda_2^F \lambda_3^F \psi \quad , \quad \psi \in P_{k-3}(F) \quad .$$

In view of Stokes' formula

$$\text{(8.128)} \qquad \underbrace{\int_F (\mathbf{q} \wedge \mathbf{n}) \cdot \mathbf{p} \ d\gamma}_{= \ 0} \ = \ \int_F \mathbf{curl}_F \varphi \cdot \mathbf{p} \ d\gamma \ = $$

$$= \int_F \varphi \ \mathrm{curl}_F \mathbf{p} \ d\gamma \ + \ \underbrace{\int_{\partial F} \varphi \ \mathbf{t} \cdot \mathbf{p} \ ds}_{= \ 0} \quad , \quad \mathbf{p} \in P_{k-2}(F)^2 \ .$$

Since the operator curl_F is surjective from $P_{k-2}(F)^2$ onto $P_{k-3}(F)$, we may choose

$$\mathrm{curl}_F \mathbf{p} \ = \ \psi \ .$$

Hence, (8.128) implies $\psi = 0$, and consequently, (8.127) gives $\mathbf{q} \wedge \mathbf{n} = 0$. □

It remains to be shown that the finite elements $(K, P_K, \Sigma_K), K \in \mathcal{T}_h$ are unisolvent, i.e., any $\mathbf{q} \in P_K := \mathbf{Nd}_k(K)$ is uniquely determined by the degrees of freedom (8.116),(8.117), and (8.118).

Theorem 8.10 Unisolvence of the edge elements
Let $\mathbf{q} \in \mathbf{Nd}_k(K), K \in \mathcal{T}_h$ and assume that

$$\int_E \mathbf{q} \cdot \mathbf{t}_E \, p_{k-1} \, ds \; = \; 0 \quad , \quad p_{k-1} \in P_{k-1}(E) \; , \; E \in \mathcal{E}_h(K) \; , \tag{8.129}$$

$$\int_F (\mathbf{q} \wedge \mathbf{n}) \cdot \mathbf{p}_{k-2} \, d\sigma \; = \; 0 \quad , \quad \mathbf{p}_{k-2} \in P_{k-2}(F)^2 \; , \; F \in \mathcal{F}_h(K) \; , \tag{8.130}$$

$$\int_K \mathbf{q} \cdot \mathbf{p}_{k-3} \, d\mathbf{x} \; = \; 0 \quad , \quad \mathbf{p}_{k-3} \in P_{k-3}(K)^3 \; . \tag{8.131}$$

Then, we have

$$\mathbf{q} \; = \; 0 \quad \text{on } K \; . \tag{8.132}$$

Proof. We will first show that (8.129)-(8.131) imply

$$\mathbf{curl}\ \mathbf{q} \; = \; 0 \quad \text{on } K \; . \tag{8.133}$$

By Green's theorem we have

$$\int_F \mathbf{grad}_F \, p \cdot (\mathbf{q} \wedge \mathbf{n}) \, d\gamma \; + \int_F p \, \mathrm{div}_F(\mathbf{q} \wedge \mathbf{n}) \, d\gamma \; = \; = \int_{\partial F} \mathbf{q} \cdot \mathbf{t} \, p \, ds \quad , \quad p \in P_{k-1}(F) \; . \tag{8.134}$$

Since $\mathbf{grad}_F \; p \; \in \; P_{k-2}(K)^2$, the first term on the left-hand side in (8.134) vanishes due to (8.130). Moreover, the boundary integral on the right-hand side in (8.134) is zero in view of (8.129). Taking further

$$\mathrm{div}_F(\mathbf{q} \wedge \mathbf{n}) \; = \; \mathbf{n} \cdot \mathbf{curl}\ \mathbf{q}$$

into account, we conclude

$$\int_F \mathbf{curl}\ \mathbf{q} \cdot \mathbf{n} \, p \, d\gamma \; = \; 0 \quad , \quad p \in P_{k-1}(F) \; .$$

Since $\mathbf{curl}\ \mathbf{q} \cdot \mathbf{n} \in P_{k-1}(F)$, it follows that

$$\mathbf{curl}\ \mathbf{q} \cdot \mathbf{n} \; = \; 0 \quad \text{on } F \; , \; F \in \mathcal{F}_h(K) \; . \tag{8.135}$$

We now use Stokes' theorem with respect to K:

$$\int_K \mathbf{q} \cdot \mathbf{curl}\ \mathbf{p}\ dx \ - \ \int_K \mathbf{p} \cdot \mathbf{curl}\ \mathbf{q}\ dx \ = \ = \int_{\partial K} (\mathbf{q} \wedge \mathbf{n}) \cdot (\mathbf{n} \wedge (\mathbf{p} \wedge \mathbf{n}))\ d\sigma \quad , \quad \mathbf{p} \in P_{k-2}(K)^3 \ . \tag{8.136}$$

Since $\mathbf{curl}\ \mathbf{p} \in P^3_{k-3}$, the first term on the left-hand side in (8.136) is zero due to (8.131), whereas the right-hand-side in (8.136) vanishes because of (8.130). Hence, we get

$$\int_K \mathbf{p} \cdot \mathbf{curl}\ \mathbf{q}\ dx \ = 0 \quad , \quad \mathbf{p} \in P_{k-2}(K)^3 \ . \tag{8.137}$$

Denoting by K_{ref} the reference tetrahedron with vertices $(0,0,0), (1,0,0), (0,1,0)$, and $(0,0,1)$ and using the affine transformation $K = F_K(K_{ref})$, for $\hat{\mathbf{q}} := \mathbf{q} \circ F_K$ we obtain by means of (8.135) and (8.137)

$$\mathbf{curl}\ \hat{\mathbf{q}} \cdot \hat{\mathbf{n}} \ = \ 0 \quad \text{on } F_{ref} \in \mathcal{F}_h(K_{ref}) \ , \tag{8.138}$$

$$\int_{K_{ref}} \mathbf{curl}\ \hat{\mathbf{q}} \cdot \hat{\mathbf{p}}\ d\hat{x} \ = \ 0 \quad , \quad \hat{\mathbf{p}} \in P_{k-2}(K_{ref})^3 \ . \tag{8.139}$$

This gives

$$\begin{aligned} (\mathbf{curl}\ \hat{\mathbf{q}})_1 \ &= \ \hat{x}_1\ \hat{\psi}_1 \quad , \quad \hat{\psi}_1 \in P_{k-2}(K_{ref}) \ , \\ (\mathbf{curl}\ \hat{\mathbf{q}})_2 \ &= \ \hat{x}_2\ \hat{\psi}_2 \quad , \quad \hat{\psi}_2 \in P_{k-2}(K_{ref}) \ , \\ (\mathbf{curl}\ \hat{\mathbf{q}})_3 \ &= \ \hat{x}_3\ \hat{\psi}_3 \quad , \quad \hat{\psi}_3 \in P_{k-2}(K_{ref}) \ . \end{aligned}$$

Finally, setting $\hat{\mathbf{p}} = (\hat{\psi}_1, \hat{\psi}_2, \hat{\psi}_3)$ in (8.139), we obtain

$$\mathbf{curl}\ \hat{\mathbf{q}} \ = \ 0 \quad ,$$

whence

$$\mathbf{curl}\ \mathbf{q} \ = \ 0 \quad .$$

Now, the last equation tells us that

$$\mathbf{q} \ = \ \mathbf{grad}\ \varphi \quad , \quad \varphi \in P_k(K) \quad .$$

By Theorem 4.5 we already know that (8.129) and (8.130) imply

$$0 \ = \ (\mathbf{q} \wedge \mathbf{n})|_F \ = \ (\mathbf{grad}\ \varphi \wedge \mathbf{n})|_F \ , \ F \in \mathcal{F}_h(K) \ .$$

Consequently, we have

$$(\mathbf{grad}\ \varphi \wedge \mathbf{n})|_F \ = \ 0 \ , \ F \in \mathcal{F}_h(K) \quad \Longrightarrow \quad \mathbf{grad}\ \varphi|_{\partial K} \ = \ const. \ .$$

Since φ is uniquely determined up to a constant, we may choose

$$\varphi|_{\partial K} \;=\; 0 \; .$$

Denoting by $\lambda_i^K, 1 \le i \le 4$, the barycentric coordinates of K, we conclude

$$\varphi \;=\; \lambda_1^K \lambda_2^K \lambda_3^K \lambda_4^K \psi \quad , \quad \psi \in P_{k-4}(K) \; . \tag{8.140}$$

By Green's formula we have

$$\int_K \varphi \, \mathrm{div}\, \mathbf{p} \, dx \;=\; - \int_K \mathbf{p} \cdot \mathbf{q} \, dx + \int_{\partial K} \varphi \, \mathbf{p} \cdot \mathbf{n} \, d\sigma \;=\; 0 \; , \; \mathbf{p} \in P_{k-3}(K)^3 \; . \tag{8.141}$$

Since the operator div is surjective from $P_{k-3}(K)^3$ onto $P_{k-4}(K)$, we may choose

$$\mathrm{div}\, \mathbf{p} \;=\; \psi \quad .$$

Hence, (8.140) and (8.141) imply $\psi = 0$ which readily gives $\mathbf{q} = 0$. □

Definition 8.6 Edge element spaces for simplicial triangulations based on edge elements of the first family

Let $\mathcal{T}_h$ be a geometrically conforming simplicial triangulation of Ω. The edge element space composed of edge elements of Nédélec's first family will be denoted by

$$\mathbf{Nd}_k(\Omega, \mathcal{T}_h) \;:=\; \{ \, \mathbf{q}_h : \bar{\Omega} \to \mathbb{R} \mid \mathbf{q}_h|_K \in \mathbf{Nd}_k(K) \; , \; K \in \mathcal{T}_h \, \} \; . \tag{8.142}$$

The construction of a basis $\mathbf{j}^{(i)}, 1 \le i \le n_h^{(k)} := \dim \mathbf{Nd}_k(\Omega; \mathcal{T}_h)$ can be done as in the case of standard finite element spaces, e.g., the Lagrangian finite element spaces.
Given a d-simplex K with $d+1$ vertices $\mathbf{x}^{(i)}, 1 \le i \le d+1$, we denote by $E_{ij} \in \mathcal{E}_h(K), 1 \le i < j \le d+1$, the edge connecting $\mathbf{x}^{(i)}$ and $\mathbf{x}^{(j)}$ and by $\mathbf{t}_{ij}$ the associated unit tangential vector pointing from $\mathbf{x}^{(i)}$ to $\mathbf{x}^{(j)}$. Further, we refer to $F_{ijk} \in \mathcal{F}_h, 1 \le i < j < k \le d+1$, as the face spanned by the vertices $\mathbf{x}^{(i)}, \mathbf{x}^{(j)}$ and $\mathbf{x}^{(k)}$.

(i) k = 1

The basis functions $\mathbf{j}(E_{ij}) = (j_1(E_{ij}), ..., j_d(E_{ij}))^T, 1 \le i < j \le d$ are defined by

$$\int_{E_{k\ell}} \mathbf{t}_{k\ell} \cdot \mathbf{j}(E_{ij}) \, ds \;=\; |E_{ij}| \; \delta_{(i,j),(k,\ell)} \quad . \tag{8.143}$$

In case $d = 2$ we get for the reference triangle K_{ref}:

$$\mathbf{j}(E_{12}) = \begin{pmatrix} 1 - x_2 \\ x_1 \end{pmatrix} \; , \; \mathbf{j}(E_{13}) = \begin{pmatrix} x_2 \\ 1 - x_1 \end{pmatrix} \; , \; \mathbf{j}(E_{23}) = \begin{pmatrix} -x_2 \\ x_1 \end{pmatrix} \; . \tag{8.144}$$

In case $d = 3$, for the reference tetrahedron K_{ref} we obtain:

$$\mathbf{j}(E_{12}) = \begin{pmatrix} 1 - x_2 - x_3 \\ x_1 \\ x_1 \end{pmatrix} \quad , \quad \mathbf{j}(E_{13}) = \begin{pmatrix} x_2 \\ 1 - x_1 - x_3 \\ x_2 \end{pmatrix} \quad , \tag{8.145}$$

$$\mathbf{j}(E_{14}) = \begin{pmatrix} x_3 \\ x_3 \\ 1 - x_1 - x_2 \end{pmatrix} \quad , \quad \mathbf{j}(E_{23}) = \begin{pmatrix} -x_2 \\ x_1 \\ 0 \end{pmatrix} \quad , \tag{8.146}$$

$$\mathbf{j}(E_{24}) = \begin{pmatrix} -x_3 \\ 0 \\ x_1 \end{pmatrix} \quad , \quad \mathbf{j}(E_{34}) = \begin{pmatrix} 0 \\ -x_3 \\ x_2 \end{pmatrix} \quad . \tag{8.147}$$

We further refer to $\mathbf{Nd}_{k,0}(\Omega; \mathcal{T}_h)$ as the subspace of $\mathbf{Nd}_k(\Omega; \mathcal{T}_h)$ with vanishing tangential trace on $\Gamma = \partial\Omega$, i.e.

$$\mathbf{Nd}_{k,0}(\Omega, \mathcal{T}_h) \ := \ \{ \ \mathbf{q} \in \mathbf{Nd}_k(\Omega, \mathcal{T}_h) \ | \ (\mathbf{q} \wedge \mathbf{n})|_\Gamma = 0 \ \} \ , \tag{8.148}$$

and to $\mathbf{Nd}_k^0(\Omega, \mathcal{T}_h)$ as the subspace of irrotational vector fields

$$\mathbf{Nd}_k^0(\Omega, \mathcal{T}_h) \ := \ \{ \ \mathbf{q} \in \mathbf{Nd}_k(\Omega, \mathcal{T}_h) \ | \ \mathbf{curl} \ \mathbf{q} = 0 \ \} \ . \tag{8.149}$$

We have the following characterization of the subspace of irrotational vector fields:

Lemma 8.3 Characterization of the subspace of irrotational vector fields

Denoting by $S_k(\Omega, \mathcal{T}_h)$ the finite element space of Lagrangian finite elements of type k, there holds:

$$\mathbf{Nd}_k^0(\Omega, \mathcal{T}_h) \ = \ \mathbf{grad} \ S_k(\Omega, \mathcal{T}_h) \ , \ \ k \in \mathbb{N} \ . \tag{8.150}$$

8.3.3 The edge elements $\mathbf{Nd}_k(K)$ of Nédélec's first family for triangulations by rectangular elements

In case of triangulations $\mathcal{T}_h$ by rectangular elements, we denote by $Q_{k_1,\ldots,k_d}(K), k_i \in \mathbb{N}_0, 1 \leq ii \leq d, K \in \mathcal{T}_h$, the linear space

$$Q_{k_1,\ldots,k_d}(K) \ := \ \{ \ \mathbf{q} : K \to \mathbb{R} \ | \ \mathbf{q} = \sum_{|\alpha_i| \leq k_i} a_\alpha \ \mathbf{x}^\alpha \ \} \ , \tag{8.151}$$

$$\dim \ Q_{k_1,\ldots,k_d}(K) \ = \ \prod_{i=1}^d (k_i + 1) \quad . \tag{8.152}$$

Definition 8.7 Edge elements for triangulations by rectangular elements

Let K be a rectangular element in $\mathbb{R}^d$ and denote by $\mathcal{E}_h(K)$ and $\mathcal{F}_h(K)$ the sets of edges resp. faces of K.
In case $d = 2$, the edge element $\mathbf{Nd}_{[k]}(K), k \in \mathbb{N}$, is defined by

$$\mathbf{Nd}_{[k]}(K) := Q_{k-1,k}(K) \times Q_{k,k-1}(K) \quad , \tag{8.153}$$

$$\dim Nd_{[k]}(K) = 2k(k+1) \quad . \tag{8.154}$$

The set Σ_K of degrees of freedom is given by

$$\int_E \mathbf{q} \cdot \mathbf{t}_E \, p \, ds \quad , \quad p \in P_{k-1}(K) \quad , \quad E \in \mathcal{E}_h(K), \tag{8.155}$$

$$\int_K \mathbf{q} \cdot \mathbf{p} \, d\mathbf{x} \quad , \quad \mathbf{p} \in Q_{k-2,k-1}(K) \times Q_{k-1,k-2}(K) \quad . \tag{8.156}$$

In case $d = 3$, the edge element $\mathbf{Nd}_{[k]}(K), k \in \mathbb{N}$, is defined by

$$\mathbf{Nd}_{[k]}(K) := Q_{k-1,k,k}(K) \times Q_{k,k-1,k}(K) \times Q_{k,k,k-1}(K) \quad , \tag{8.157}$$

$$\mathbf{Nd}_{[k]}(K) = 3k(k+1)^2 \quad . \tag{8.158}$$

The set Σ_K of degrees of freedom is given by

$$\int_E \mathbf{q} \cdot \mathbf{t}_E \, p \, ds \ , \ p \in P_{k-1}(E), E \in \mathcal{E}_h(K), \tag{8.159}$$

$$\int_F (\mathbf{q} \wedge \mathbf{n}_F) \cdot \mathbf{p} \, d\mathbf{x} \ , \ \mathbf{p} \in Q_{k-2,k-1}(F) \times Q_{k-1,k-2}(F), F \in \mathcal{F}_h(K), \tag{8.160}$$

$$\int_K \mathbf{q} \cdot \mathbf{p} \, d\mathbf{x} \ , \ \mathbf{p} \in Q_{k-1,k-2,k-2}(K) \times Q_{k-2,k-1,k-2}(K) \times Q_{k-2,k-2,k-1}(K). \tag{8.161}$$

Definition 8.8 Edge element spaces based on triangulations by rectangular elements

Let $\mathcal{T}_h$ be a geometrically conforming triangulation of a bounded domain $\Omega \subset \mathbb{R}^d$. The edge element space $\mathbf{Nd}_{[k]}(\Omega, \mathcal{T}_h), k \in \mathbb{N}$, is defined as follows

$$\mathbf{Nd}_{[k]}(\Omega, \mathcal{T}_h) := \{ \mathbf{q} : \bar{\Omega} \to \mathbb{R} \mid \mathbf{q}|_K \in Nd_{[k]}(K) \ , \ K \in \mathcal{T}_h \ \} \ . \tag{8.162}$$

Theorem 8.11 Unisolvence of the edge elements for rectangular elements

Let K be a rectangular element in $\mathbb{R}^d$ and let the set of degrees of freedom be given by (8.157),(8.158) resp. (8.159),(8.160),(8.161). Then the edge element $(K, \mathbf{Nd}_{[k]}(K), \Sigma_K)$ is unisolvent.

Proof. The proof is left as an exercise. □

Theorem 8.12 $\mathbf{H}(\mathbf{curl};\Omega)$-conformity of $\mathbf{Nd}_{[k]}(\Omega,\mathcal{T}_h)$
The edge element spaces $\mathbf{N}bd_{[k]}(\Omega,\mathcal{T}_h)$, $k \in \mathbb{N}$, are $\mathbf{H}(\mathbf{curl};\Omega)$-conform, i.e.,

$$\mathbf{Nd}_{[k]}(\Omega,\mathcal{T}_h) \subset \mathbf{H}(\mathbf{curl};\Omega) \ , \ k \in \mathbb{N} \ . \tag{8.163}$$

Proof. The proof is left as an exercise. □

A basis $\mathbf{j}^{(i)}, 1 \leq i \leq n_h^{(k)} := \dim \mathbf{Nd}_{[k]}(\Omega,\mathcal{T}_h)$ can be constructed following the same lines as in the subsection before.
Given a d-rectangle K with 2^d vertices $\mathbf{x}^{(i)}, 1 \leq i \leq 2^d$,, counted from left to right and bottom to top, we denote by $E_{ij} \in \mathcal{E}_h(K)$ the edge connecting $\mathbf{x}^{(i)}$ and $\mathbf{x}^{(j)}$ and by $\mathbf{t}_{ij}$ the associated unit tangential vector pointing from $\mathbf{x}^{(i)}$ to $\mathbf{x}^{(j)}$.
In case $k = 1$, the basis functions $\mathbf{j}(E_{ij}) = (j_1(E_{ij}),...,j_d(E_{ij})^T, E_{ij} \in \mathcal{E}_h(K)$, are defined by

$$\int_{E_{k\ell}} \mathbf{t}_{k\ell} \cdot \mathbf{j}(E_{ij}) \ ds \ = \ |E_{ij}| \ \delta_{(i,j),(k,\ell)} \quad . \tag{8.164}$$

In case $d = 2$ we get for the reference rectangle K_{ref}:

$$\mathbf{j}(E_{12}) = \begin{pmatrix} 1 - x_2 \\ 0 \end{pmatrix} \ , \ \mathbf{j}(E_{34}) = \begin{pmatrix} x_2 \\ 0 \end{pmatrix} \ , \tag{8.165}$$

$$\mathbf{j}(E_{14}) = \begin{pmatrix} 0 \\ 1 - x_1 \end{pmatrix} \ , \ \mathbf{j}(E_{23}) = \begin{pmatrix} 0 \\ x_1 \end{pmatrix} \ . \tag{8.166}$$

The case $d = 3$ is left as an exercise.

References

[1] F. Brezzi and M. Fortin, Mixed and hybrid finite element methods. Springer Series in Computational Mathematics, Springer-Verlag, New York, 1991
[2] V. Girault and P.A. Raviart, Finite Element Approximation of the Navier-Stokes Equations. Lecture Notes in Mathematics **749**, Springer, Berlin-Heidelberg-New York, 1979
[3] J. Hiriart-Urruty and C. Lémaréchal, Convex Analysis and Minimization Algorithms. Springer, Berlin-Heidelberg-New York, 1993
[4] P.A. Raviart and J.M. Thomas, A mixed finite element method for second order elliptic problems. In: Mathematical Aspects of the Finite Element Method (I. Galligani and E. Magenes; eds.), Lecture Notes in Mathematics **606**, Springer, Berlin-Heidelberg-New York, 1977
[5] K. Yosida, Functional Analysis. Springer, New York, 1966