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Numerical Analysis II, Spring 2010

Numerical Analysis II (Homework 2)

Exercise 4 (*Initial value problem with discontinuous right-hand side*)

Consider the initial value problem

$$\begin{aligned} y'(x) &= -\operatorname{sgn}(y(x)) \ , \ x \geq -1, \\ y(-1) &= 1 \ . \end{aligned}$$

(i) Compute the exact solution of the differential equation. Here, exact solution means a piecewise continuously differentiable function which satisfies the differential equation almost everywhere.
(ii) For $x_i := -1 + ih, i \in \mathbb{N}_0$, and $0 < h < 1$ consider the explicit Euler method

$$y_{i+1} = y_i - h \operatorname{sgn}(y_i) \quad , \quad y_0 = 1 \ .$$

Show that for $n \in \mathbb{N}$ with $nh \leq 1 < (n+1)h$ and all $k \in \mathbb{N}$ there holds

$$y_{n+2k} = y_n \quad , \quad y_{n+2k+1} = y_{n+1} \ .$$

(iii) Give a reason for the oscillating behavior of the approximations. How can you avoid the oscillations?

6 Points

Exercise 5 (*Midpoint rule applied to a model problem*)

For the numerical integration of the model problem

$$y'(x) = \lambda \, y(x) \ , \ x \geq a \quad , \quad y(a) = \alpha$$

consider the explicit midpoint rule with an initial Euler step using an equidistant grid $x_i := a + ih, h > 0$:

$$\begin{aligned} y_{i+1} &= y_{i-1} + 2h\lambda y_i) \ , \ i \geq 1 \ , \\ y_1 &= y_0 + h\lambda y_0 \ , \\ y_0 &= \alpha \ . \end{aligned}$$

(i) Determine functions $g_n(\lambda h), 1 \leq n \leq 2$, such that the sequences

$$y_i^{(n)} := g_n^i(\lambda h) y_0$$

satisfy the difference equation associated with the explicit midpoint rule.

(ii) Assume $\lambda \ll 0$. Does the approximate solution show the same qualitative behavior as the exact solution?

(iii) Determine $\mu_k, 1 \leq k \leq 2$, depending on λ and h, such that

$$y_i \;=\; (\mu_1 g_1^i + \mu_2 g_2^i)\ \alpha\ .$$

Derive a modification of the Euler initial step to improve the quality of the method in case $\lambda \ll 0$.

6 Points

Exercise 6 (*Lady Windermere's fan*)

Assume that $f : I \times D \to D, I := [a,b] \subset \mathbb{R}, D \subset \mathbb{R}^m$, is continuous on $I \times D$ and satisfies the Lipschitz condition

$$\|f(x,y_1) - f(x,y_2)\| \;\leq\; L\ \|y_1 - y_2\|\ ,\ x \in I\ ,\ y_1,y_2 \in D\ ,\ L > 0\ .$$

For the numerical integration of the initial value problem

$$\begin{aligned} y'(x) &= f(x,y(x))\ ,\ x \in I := [a,b] \subset \mathbb{R}\ , \\ y(a) &= \alpha \in D \end{aligned}$$

consider the explicit one-step method

$$\begin{aligned} y_{i+1} &= y_i + h_i\ \Phi(x_i, y_i, h_i; f)\ , \\ y_0 &= \alpha\ , \end{aligned}$$

where $a = x_0 < x_1 < ... < x_N = b$ is a not necessarily equidistant partition of I with step sizes $h_i := x_{i+1} - x_i, 0 \leq i \leq N-1$.
Prove the following assertion: If the one-step method is consistent of order $p \in \mathbb{N}$, i.e.,

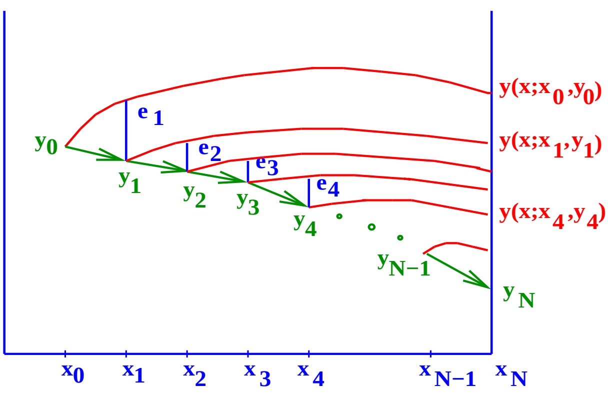
$$\|\tau_h(x,y)\| \;\leq\; C h_{max}^p\ ,\ x \in I\ ,\ y \in D\ ,$$

where $h_{max} := \max\limits_{0 \leq i \leq n-1} (x_{i+1} - x_i)$ and $C \in \mathbb{R}_+$ is independent of h_{max}, then there exists a constant $C' \in \mathbb{R}_+$, independent of h_{max}, such that

$$\|y_N - y(b)\| \;\leq\; C'\ \frac{\exp(L(b-a)-1)}{L}\ h_{max}^p\ .$$

6 Points

[Hint: Use the technique known as Lady Windermere's fan which is illustrated in the figure below:



Construct N initial value problems of the form

$$\begin{aligned} y'(x) &= f(x,y(x)) \ , \ x \geq x_i \ , \ 0 \leq i \leq N-1 \ , \\ y(x_i) &= y_i \end{aligned}$$

with the solutions $y(x;x_i,y_i)$. The quantities $e_i := \|y(x_i;x_{i-1},y_{i-1}) - y_i\|$ are available by means of the local discretization error. Interpreting $y(x;x_i,y_i)$ as the solution of the perturbed initial value problem

$$\begin{aligned} y'(x) &= f(x,y(x)) \ , \ x \geq x_i \ , \ 0 \leq i \leq N-1 \ , \\ y(x_{i+1}) &= y_{i+1} + \underbrace{h_i \tau_{h_i}(x_i,y_i)}_{perturbation} \ , \end{aligned}$$

the error propagation can be accessed by Gronwall's lemma.]

Delivery of the homework at latest on February 22, 2010. The homework may be submitted either electronically (rohop@math.uh.edu) or as a hardcopy in class.