University of Houston
Department of Mathematics
Dr. Ronald H.W. Hoppe
Numerical Analysis II, Spring 2010

Numerical Analysis II (Homework 4)

Exercise 10 (*Consistency of linear multi-step methods*)

Assume $f \in C^p(I \times D), I := [a,b] \subset \mathbb{R}, D \subset \mathbb{R}^d$ and $\alpha \in \mathbb{R}^d$. Consider the initial-value problem

$$
\begin{aligned}
(IVP_1) \qquad y'(x) &= f(x, y(x)) \quad , \quad x \in I \ , \\
(IVP_2) \qquad y(a) &= \alpha \ .
\end{aligned}
$$

For a given equidistant partition $x_j := a + jh, h := (b-a)/N, N \in \mathbb{N}$, of I and given real numbers $\alpha_k, \beta_k, 0 \leq k \leq m, m \in \mathbb{N}$, with $\alpha_m \neq 0$ and vectors $\alpha^{(j)} \in \mathbb{R}^d, 0 \leq j \leq m-1$, a linear multi-step method is of the form

$$
\begin{aligned}
(MSM_1) \qquad \frac{1}{h} \sum_{k=0}^{m} \alpha_k \ y_{j+k} &= \sum_{k=0}^{m} \beta_k \ f(x_{j+k}, y_{j+k}) \quad , \quad 0 \leq j \leq N-m \ , \\
(MSM_2) \qquad y_j &= \alpha^{(j)} \quad , \quad 0 \leq j \leq m-1 \ .
\end{aligned}
$$

Denoting by τ_h the local discretization error, suppose that

$$
(\star) \qquad \tau_h(x_j) = O(h^p) \quad , \quad 0 \leq j \leq m-1 \ .
$$

Show that the linear multi-step method $(MSM)_1, (MSM)_2$ is consistent with the initial-value problem $(IVP)_1, (IVP)_2$ of order p if and only if it is consistent of order p with

$$
\begin{aligned}
(IVP_1') \qquad y'(x) &= y(x) \quad , \quad x \in I \ , \\
(IVP_2') \qquad y(a) &= 1 \ .
\end{aligned}
$$

6 Points

Exercise 11 (*Stability of multi-step methods*)

In class, the stability of multi-step methods has been investigated by means of the Lipschitz stability of operators on the linear space of grid functions. This approach can be generalized as follows:
Let $(X, \|\cdot\|_X)$ and $(Y, \|\cdot\|_Y)$ be normed spaces and let $(A_h)_h$ be a family of continuous operators $A_h : X \to Y$. This family is called Lipschitz stable, if there exist positive numbers δ, η such that for all $x, z \in X$ and all h satisfying

$$
(LS)_1 \qquad \|A_h x - A_h z\|_Y \ \leq \ \delta
$$

there holds

$$(LS)_2 \qquad \|x - z\|_X \;\le\; \eta\, \|A_h x - A_h z\|_Y \;.$$

(i) For Lipschitz stable families of continuous linear operators $A_h : X \to Y$ derive a simpler definition of stability by taking advantage of the linearity.

(ii) Let $(A_h)_h$ be a Lipschitz stable family of continuous linear operators which are additionally assumed to be surjective.
Show that for all h the operator A_h is invertible with $\|A_h^{-1}\| \le C, C > 0$, and derive an upper bound for the constant C.
(Hint: Show injectivity first and then use the definition of the operator norm $\|A_h\| = \sup\limits_{0 \neq x \in X} \frac{\|A_h x\|_Y}{\|x\|_X}$.)

(iii) Let H be a Hilbert space, i.e., a complete inner product space with inner product $< \cdot, \cdot >: H \times H \to \mathbb{R}$, and let $(A_h)_h$ be a family of continuous endomorphisms $A_h : H \to H$. Further, let $a_h : H \times H \to \mathbb{R}$ be the bilinear form given by $a_h(x, y) :=< A_h x, y >$.
Prove the equivalence of the following two statements:

($*$) For all h the operator A_h is invertible such that $\|A_h^{-1}\| \le C$, where C is a positive constant independent of h.

($**$) There holds

$$\begin{aligned} \sup_{\|y\|=1} |a_h(x, y)| \;&>\; \frac{1}{C}\, \|x\| \quad , \quad x \in H \;, \\ \sup_{\|x\|=1} |a_h(x, y)| \;&>\; 0 \quad , \quad 0 \neq y \in H \;. \end{aligned}$$

(Hint: For ($*$) $\Longrightarrow$ ($**$) use $\|x\| = \sup\{< x, y > \;\mid\; \|y\| = 1\}, x \in H$. For ($**$) $\Longrightarrow$ ($*$) verify first that $\|A_h x\| \ge C^{-1}\|x\|, x \in H$. Then, using an arbitrary Cauchy sequence, show that the range of each operator A_h is closed. Finally, prove surjectivity by contradiction, i.e., assume range$(A_h)^\perp \neq \{0\}$.)

6 Points

Delivery of the homework at latest on March 29, 2010. The homework may be submitted either electronically (rohop@math.uh.edu) or as a hardcopy in class.