Department of Mathematics
University of Houston
Numerical Analysis II
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Numerical Mathematics II
(8th Homework Assignment)

Exercise 20 (*Stability: functional analytic investigation*)

In class, the stability theory of multi-step methods has been introduced on the basis of the notion of Lipschitz stability of operators on the linear space of grid functions. This theory can be generalized and put into a functional analytic framework:
Let $(X, \|\cdot\|_X)$ and $(Y, \|\cdot\|_Y)$ be normed spaces and $(A_h)_h$ a family of continuous operators $A_h : X \to Y$ indexed by h. This family is said to be Lipschitz stable, if there exist positive numbers δ, η such that for all $x, z \in X$ and all h satisfying

$$(LS)_1 \qquad \|A_h x - A_h z\|_Y \ \leq \ \delta$$

there holds

$$(LS)_2 \qquad \|x - z\|_X \ \leq \ \eta \ \|A_h x - A_h z\|_Y \ .$$

The number η is called stability bound, whereas δ is referred to as the stability threshold.

(i) Derive a much simpler definition of Lipschitz-stable families of continuous linear maps $A_h : X \to Y$ by taking advantage of the linearity.

(ii) Assume $(A_h)_h$ to be a family of continuous, Lipschitz-stable linear mappings and suppose in addition that the maps are surjective.
Show that A_h is invertible for all h with $\|A_h^{-1}\| \leq C, C > 0$, and provide an upper bound for the constant C.
(Hint: Firstly, verify injectivity and secondly, take advantage of the definition of the operator norm $\|A_h\| = \sup\limits_{0 \neq x \in X} \frac{\|A_h x\|_Y}{\|x\|_X}$.)

(iii) Assume H to be a Hilbert space over $\mathbb{R}$, i.e., a complete normed space with inner product $< \cdot, \cdot >: H \times H \to \mathbb{R}$, and assume that $(A_h)_h$ is a family

of continuous endomorphisms $A_h : H \to H$. Moreover, define a bilinear form $a_h : H \times H \to \mathbb{R}$ according to $a_h(x,y) :=< A_h x, y >$.
Prove the equivalence of the following two statements:

$(*)$ A_h is invertible with $\|A_h^{-1}\| \leq C$ for all h, where C is a positive constant independent of h.

$(**)$ There holds

$$\sup_{\|y\|=1} |a_h(x,y)| \; > \; \frac{1}{C} \, \|x\| \quad , \quad x \in H \; ,$$
$$\sup_{\|x\|=1} |a_h(x,y)| \; > \; 0 \quad , \quad 0 \neq y \in H \; .$$

(Hint: For $(*) \Longrightarrow (**)$ use the relation $\|x\| = \sup\{< x, y > | \|y\| = 1\}, x \in H$. For $(**) \Longrightarrow (+)$ show first that $\|A_h x\| \geq C^{-1}\|x\|, x \in H$. Then, using an arbitrary Cauchy sequence, show that the range of each operator A_h is closed. Finally, prove surjectivity by contradiction assuming range$(A_h)^\perp \neq \{0\}$.)

6 Points

<u>**Exercise 21**</u> (*Adams-Basforth method*)

For a scalar initial value problem of the form

$$y'(x) \; = \; f(x, y(x)) \quad , \quad y(a) \; = \; \alpha \; ,$$

Adams-Bashforth m-step methods can be described by

$$y_{j+m} \; = \; y_{j+m-1} \; + \int\limits_{x_{j+m-1}}^{x_{j+m}} p(x) \; dx \quad ,$$

where $p \in P_{m-1}$ is the interpolating polynomial given by $p(x_k) = f(x_k, y_k), j \leq k \leq j+m-1$. In the sequel, assume an equidistant grid of step size $h > 0$.

(i) Prove the stability of the m-step Adams-Bashforth method.

(ii) Determine the order of consistency of the m-step Adams-Bashforth method.

(iii) Using part (ii) of Exercise 18, prove the following representation of the m-step Adams-Bashforth method on the basis of backward differences:

$$y_{j+m} \; = \; y_{j+m-1} \; + \; h \sum_{k=0}^{m-1} \gamma_k \; \nabla^k f_{j+m-1} \quad , \quad 0 \leq j \leq N-m \quad ,$$

where

$$\gamma_k \; = \; (-1)^k \int\limits_0^1 \binom{-s}{k} \; ds \quad .$$

What is the advantage of this representation with respect to an order control?

(iv) Prove the following property of the coefficients γ_k from part (iii)

$$\sum_{\ell=0}^{k} \frac{1}{k - \ell + 1} \gamma_\ell = 1 \quad .$$

(Hint: Consider a power series with coefficients γ_k and use the Cauchy product of power series.)

(v) Use an algorithmic notation to derive an efficient implementation of an Adams-Bashforth method with order control where the number $p_k \in \mathbb{N}$ describes the order in the k-th step. Assume that the sequence $(p_k)_{k=0}^{N-p_0}$ is given and the initial values $y_0, ..., y_{p_0-1}$ are known.

10 Points

Delivery of the homework at latest on April 11, 2006. The homework may be submitted either electronically (rohop@math.uh.edu) or as a hardcopy in class.