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Department of Mathematics
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Numerical Partial Differential Equations

Numerical Partial Differential Equations (Homework 2)

Exercise 2 (*Compact Finite Difference Approximation of the Laplacian*)

Show that it is not possible to obtain a compact finite difference approximation of the Laplacian of order $O(h^3)$.

4 Points

Exercise 3 (*Higher Order Finite Difference Approximation*)

Let $\Omega := (a,b)^2, a, b \in \mathbb{R}, a < b$, and assume that $\overline{\Omega}_h$ is a uniform grid of step size $h > 0$. For the approximation of Poisson's equation with homogeneous Dirichlet boundary conditions consider the finite difference approximation

$$\begin{aligned}
&-\frac{1}{6}\Big(u_h(x_1+h,x_2+h) + u_h(x_1+h,x_2-h) + u_h(x_1-h,x_2+h) + \\
&+u_h(x_1-h,x_2-h)\Big) \ + \ \frac{2}{3}\Big(u_h(x_1+h,x_2) + u_h(x_1-h,x_2) + \\
&+u_h(x_1,x_2+h) + u_h(x_1,x_2-h)\Big) \ + \ \frac{10}{3}\, u_h(x_1,x_2) \ = \\
&= \ \frac{h^2}{12}\Big(f(x_1+h,x_2) + f(x_1-h,x_2) + f(x_1,x_2+h) + \\
&\qquad +f(x_1,x_2-h) + 8f(x_1,x_2)\Big) \quad , \quad (x_1,x_2) \in \Omega_h \ , \\
&u_h(x) \ = \ 0 \quad , \quad (x_1,x_2) \in \Gamma_h \ ,
\end{aligned}$$

and show that for $u \in C^6(\overline{\Omega})$ it is consistent of order $O(h^4)$.

4 Points

Exercise 4 (*Comparison of Finite Difference Approximations*)

For the finite difference approximation of Poisson's equation

$$\begin{aligned} -\Delta u &= f \quad \text{in } \Omega = (0,1)^2 \ , \\ u &= 0 \quad \text{on } \Gamma = \partial\Omega \ , \end{aligned}$$

with respect to uniform grids $\overline{\Omega}_{h_\nu}$ of step sizes $h_\nu = 1/(N_\nu + 1)$ use

(i) the five-point difference approximation of the Laplacian,

(ii) the nine-point difference approximation of the Laplacian,

(iii) the finite difference approximation from Exercise 3,

in case f is chosen such that $u(x_1, x_2) = \sin(\pi x_1) \sin(\pi x_2)$ is the exact solution. For $h_\nu = 2^{-(\nu+2)}, 1 \leq \nu \leq 4$, compute

$$e_\nu \ := \ \max_{x \in \overline{\Omega}_h} |u(x) - u_h(x)| \quad , \quad 1 \leq \nu \leq 4 \ ,$$

and

$$r_\nu \ := \ \frac{\log(e_{\nu-1}/e_\nu)}{\log(h_{\nu-1}/h_\nu)} \quad , \quad 2 \leq \nu \leq 4 \ .$$

Show the results in form of a table

	Five-Point		Nine-Point		FD from Ex. 3	
h_ν	e_ν	r_ν	e_ν	r_ν	e_ν	r_ν
$\frac{1}{8}$						
$\frac{1}{16}$						
$\frac{1}{32}$						
$\frac{1}{64}$						

Display e_ν in the form $x.xxE - xx$ and r_ν in the form $x.xx$.

6 Points

Exercises 2 and 3 are due on February 6, 2008. Exercise 4 is due on February 13, 2008. The homework may be submitted either electronically (rohop@math.uh.edu) or as a hardcopy in class