1. For each matrix, determine its two eigenvalues, λ_1 and λ_2, and associated eigenvectors, $\mathbf{r}_1$ and $\mathbf{r}_2$.

(a) $\begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix}$ (b) $\begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix}$ (c) $\begin{pmatrix} 1 & 2 \\ 2 & 2 \end{pmatrix}$ (d) $\begin{pmatrix} 0 & 2 \\ -2 & 0 \end{pmatrix}$

(Answer for (d): $\lambda_1 = -2i$, $\lambda_2 = 2i$, $\mathbf{r}_1 = (i, 1)^T$ and $\mathbf{r}_2 = (i, -1)^T$.)

2. Define two 2×2 matrices Λ and R as follows. Let Λ be a diagonal matrix with diagonal entries λ_1 and λ_2. Let R be a 2×2 matrix with columns which are 2-vectors $\mathbf{r}_1$ and $\mathbf{r}_2$. For each matrix in the previous exercise, call each A, let λ_1, λ_2 be its eigenvalues, and let $\mathbf{r}_1$, $\mathbf{r}_2$ be the associated eigenvectors. Write down Λ and R and compute $R^{-1}AR$ and $R\Lambda R^{-1}$.

3. This exercise will use the eigenvalues, λ_1 and λ_2, and eigenvectors, $\mathbf{r}_1$ and $\mathbf{r}_2$, you computed for the matrix, say A, from exercise 1(d) above. Define the following diagonal matrix

$$\exp(\Lambda t) \equiv \begin{pmatrix} e^{\lambda_1 t} & 0 \\ 0 & e^{\lambda_2 t} \end{pmatrix}, \text{ where } t \text{ is a real variable,}$$

and let R be the 2×2 matrix with columns $\mathbf{r}_1$ and $\mathbf{r}_2$.

(a) Explicitly compute the matrix $\exp(At) \equiv R\, \exp(\Lambda t)\, R^{-1}$.

(b) Compute the derivative $\dfrac{d}{dt} \exp(At)$. (Your answer should be $A \exp(At)$.)

It's a fun exercise to rewrite $\exp(At)$ using Euler's formula $e^{i\theta} = \cos\theta + i \sin\theta$. This will give you a mechanism to eliminate the i's from the formula for $\exp(At)$ you just computed. Give this a try only if you have time.

4. Use the previous exercise to find the general solution to the <u>coupled</u> system of differential equations $\dfrac{dx}{dt} = 2y$, $\dfrac{dy}{dt} = -2x$. (Hint: $(x(t), y(t))^T = \exp(At)(x_0, y_0)^T$ where x_0 and y_0 are scalar constants.)

5. An $n \times n$ matrix is said to be *diagonalizable* when the set of its eigenvectors forms an n-dimensional basis. A given matrix may or may not be diagonalizable. From the following 2×2 matrices, determine which are diagonalizable.

(a) $\begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix}$ (b) $\begin{pmatrix} -1 & 4 \\ -1 & 3 \end{pmatrix}$ (c) $\begin{pmatrix} -2 & 4 \\ -1 & 2 \end{pmatrix}$ (d) $\begin{pmatrix} 0 & 2 \\ 2 & 0 \end{pmatrix}$

6. Determine all eigenvalues and eigenvectors for the following.

(a) $\begin{pmatrix} 4 & -1 & -1 \\ -2 & 5 & -1 \\ -2 & 1 & 3 \end{pmatrix}$ (b) $\begin{pmatrix} 2 & -1 & 1 \\ 0 & 1 & 1 \\ 0 & 1 & 1 \end{pmatrix}$ (c) $\begin{pmatrix} 1 & -1 & 2 \\ 1 & 1 & 0 \\ 1 & 1 & 0 \end{pmatrix}$

(BTW: Two of these are diagonalizable, one is not.)