Math 3363 Sanders Spring 2008

Some Homework Questions for Exam 3

1. Derive the 2π periodic Fourier series for the following functions.

(a) $f(\theta) = \theta$ (b) $f(\theta) = \theta^2$

Use these results and Parseval to determine values for the following.

(c) $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^2}$ (d) $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^4}$

2. Use the chain rule, together with the change of variables $x = r\cos(\theta)$, $y = r\sin(\theta)$, to derive

$$\nabla^2 u = \frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r} + \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2}$$

3. Solve Laplace's equation $\nabla^2 u = 0$ on the unit disk $r < 1$ subject to the following boundary conditions at $r = 1$.

(a) $u(1,\theta) = 1 + 2\cos(3\theta) + 3\sin(4\theta)$ (b) $\dfrac{\partial}{\partial r}u(1,\theta) + u(1,\theta) = \cos(2\theta)$

4. Solve Laplace's equation $\nabla^2 u = 0$ on the annular region $1 < r < 2$ with boundary conditions at $r = 1$ and $r = 2$ given as follows.

(a) $u(1,\theta) = 2,\ u(2,\theta) = 1$ (b) $\dfrac{\partial}{\partial r}u(1,\theta) = 0,\ u(2,\theta) = \cos(\theta)$

5. The 2π periodic Fourier series can be written in terms of complex exponentials as

$$f(\theta) = \sum_{n=-\infty}^{\infty} a_n e^{in\theta}, \text{ where } a_n = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(\phi)e^{-in\phi}\,d\phi.$$

(a) Use this to derive a general solution to Laplace's equation in polar coordinates given by

$$u(r,\theta) = a_0 + \tilde{a}_0 \ln(r) + \sum_{n\neq 0}(a_n r^n + \tilde{a}_n r^{-n})e^{in\theta}.$$

(b) Given that $u(r,\theta)$ is nonsingular at $r = 0$, show that this solution reduces to

$$u(r,\theta) = \sum_{n=-\infty}^{\infty} b_n r^{|n|} e^{in\theta}.$$

(c) Show the solution to Laplace's equation $\nabla^2 u = 0$ on $r < 1$ with boundary condition $u(1,\theta) = f(\theta)$ is given by

$$u(r,\theta) = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(\phi)\left(\sum_{n=-\infty}^{\infty} r^{|n|}e^{in(\theta-\phi)}\right)d\phi.$$

(d) Use the formula for the geometric series $\sum_{n=0}^{\infty}\rho^n = 1/(1-\rho)$ (with $|\rho| < 1$) to derive the following closed form expression for the infinite series

$$\sum_{n=-\infty}^{\infty} r^{|n|}e^{in(\theta-\phi)} = \frac{1-r^2}{1-2r\cos(\theta-\phi)+r^2} \quad \text{(valid for } |r| < 1.)$$

(Don't expect this, #5, on exam 3. I may put it on the final however.)

6a. Use a solution $u(z)$ to $\dfrac{d^2u}{dz^2} + z^2 \dfrac{du}{dz} = 0$ to determine a solution to $\dfrac{d^2v}{dx^2} + 5x^2 \dfrac{dv}{dx} = 0$. (Hint: Let $z = \alpha x$ for an appropriately chosen constant α.)

6b. Suppose you know the solution $u(r, \theta)$ to $\nabla^2 u = 0$ on the unit disk $r < 1$ with boundary condition $u(1, \theta) = f(\theta)$. Find the solution to $\nabla^2 v = 0$ on the disk $r < R$ with boundary condition $v(1, R) = f(\theta)$ in terms of u. (Answer: $v(r, \theta) = u(r/R, \theta)$.)

7. Use a change of variables to derive all eigenvalues and eigenfunctions to $u_{xx} = \lambda u$ on the interval (a, b) subject to boundary conditions

(a) $u(a) = 0,\ u(b) = 0.$ (b) $u_x(a) = 0,\ u_x(b) = 0.$

(You are allowed to assume your known results when $a = 0$ and $b = 1$.)

8. Use the Fourier transform technique to solve the following first order equations on the infinite interval $-\infty < x < \infty$.

(a) $\begin{cases} \dfrac{\partial u}{\partial t} + 4\dfrac{\partial u}{\partial x} = 0 \\ u(x,0) = f(x). \end{cases}$ (b) $\begin{cases} \dfrac{\partial u}{\partial t} + \dfrac{\partial u}{\partial x} + 2u = 0 \\ u(x,0) = f(x). \end{cases}$

(Your final answer must not involve the Fourier transform of f.)

9. Use the Fourier transform technique to solve the second order wave equation on the infinite interval $-\infty < x < \infty$.

$$\frac{\partial^2 u}{\partial t^2} - \frac{\partial^2 u}{\partial x^2} = 0 \quad \begin{matrix} u(x,0) = f(x) \\ u_t(x,0) = 0 \end{matrix}.$$

(Answer: $u(x,t) = \frac{1}{2}(f(x+t) + f(x-t))$.)

10. Use the Fourier transform technique to solve the second order wave equation on the infinite interval $-\infty < x < \infty$.

$$\frac{\partial^2 u}{\partial t^2} - \frac{\partial^2 u}{\partial x^2} = 0 \quad \begin{matrix} u(x,0) = 0 \\ u_t(x,0) = g(x) \end{matrix}.$$

(Answer: $u(x,t) = \frac{1}{2}\int_{x-t}^{x+t} g(y)\, dy$. Don't expect this, #10, on exam 3. I may put it on the final however.)