①

Power Series of Matrices

1) $A = \begin{pmatrix} -1 & 6 \\ -1 & 4 \end{pmatrix}$ $R = \begin{pmatrix} 3 & 2 \\ 1 & 1 \end{pmatrix}$ $\Lambda = \begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix}$

$R^{-1} = \frac{1}{3-2}\begin{pmatrix} 1 & -2 \\ -1 & 3 \end{pmatrix}$ (I used 2x2 Cramer's rule trick)

So

a) $\cos(At) = R\cos(\Lambda t)R^{-1}$

$$= \begin{pmatrix} 3 & 2 \\ 1 & 1 \end{pmatrix}\begin{pmatrix} \cos(1t) & 0 \\ 0 & \cos(2t) \end{pmatrix}\begin{pmatrix} 1 & -2 \\ -1 & 3 \end{pmatrix}$$

$$= \begin{pmatrix} 3 & 2 \\ 1 & 1 \end{pmatrix}\begin{pmatrix} \cos(t) & -2\cos(t) \\ -\cos(2t) & 3\cos(2t) \end{pmatrix}$$

$$\boxed{\cos(At) = \begin{pmatrix} 3\cos(t) - 2\cos(2t) & -6\cos(t) + 6\cos(2t) \\ \cos(t) - \cos(2t) & -2\cos(t) + 3\cos(2t) \end{pmatrix}}$$

b) $\sin(At) = R\sin(\Lambda t)R^{-1}$

$$= \begin{pmatrix} 3 & 2 \\ 1 & 1 \end{pmatrix}\begin{pmatrix} \sin(t) & 0 \\ 0 & \sin(2t) \end{pmatrix}\begin{pmatrix} 1 & -2 \\ -1 & 3 \end{pmatrix}$$

$$\boxed{\sin(At) = \begin{pmatrix} 3\sin(t) - 2\sin(2t) & -6\sin(t) + 6\sin(2t) \\ \sin(t) - \sin(2t) & -2\sin(t) + 3\sin(2t) \end{pmatrix}}$$

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c) $\frac{d}{dt}\sin(At) = \begin{pmatrix} 3\cos t - 4\cos(2t) & -6\cos t + 12\cos(2t) \\ \cos(t) - 2\cos(2t) & -2\cos t + 6\cos(2t) \end{pmatrix}$

$A\cos(At)$

$$= \begin{pmatrix} -1 & 6 \\ -1 & 4 \end{pmatrix}\begin{pmatrix} 3\cos(t) - 2\cos(2t) & -6\cos(t) + 6\cos(2t) \\ \cos(t) - \cos(2t) & -2\cos(t) + 3\cos(2t) \end{pmatrix}$$

$$= \begin{pmatrix} -3\cos(t) + 2\cos(2t) + 6\cos(t) - 6\cos(2t) & 6\cos(t) - 6\cos(2t) - 12\cos(2t) + 18\cos(2t) \\ -3\cos(t) + 2\cos(2t) + 4\cos(t) - 4\cos(2t) & 6\cos(t) - 6\cos(2t) - 8\cos(t) + 12\cos(2t) \end{pmatrix}$$

$$= \begin{pmatrix} 3\cos(t) - 4\cos(2t) & -6\cos(t) + 12\cos(2t) \\ \cos(t) - 2\cos(2t) & -2\cos(t) + 6\cos(2t) \end{pmatrix}$$

$$= \frac{d}{dt}\sin(At) \checkmark$$

2) $A = \begin{pmatrix} -1 & 6 \\ -1 & 4 \end{pmatrix}$ $R = \begin{pmatrix} 3 & 2 \\ 1 & 1 \end{pmatrix}$ $R^{-1} = \begin{pmatrix} 1 & -2 \\ -1 & 3 \end{pmatrix}$

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a) $e^{At} = R e^{\Lambda t} R^{-1}$

$$= \begin{pmatrix} 3 & 2 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} e^{t} & 0 \\ 0 & e^{2t} \end{pmatrix} \begin{pmatrix} 1 & -2 \\ -1 & 3 \end{pmatrix}$$

$$= \begin{pmatrix} 3e^{t} - 2e^{2t} & -6e^{t} + 6e^{2t} \\ e^{t} - e^{2t} & -2e^{t} + 3e^{2t} \end{pmatrix}$$

b) $$\frac{d}{dt} e^{At} = \begin{pmatrix} 3e^{t} - 4e^{2t} & -6e^{t} + 12e^{2t} \\ e^{t} - 2e^{2t} & -2e^{t} + 6e^{2t} \end{pmatrix}$$

$$Ae^{At} = \begin{pmatrix} -1 & 6 \\ -1 & 4 \end{pmatrix} \begin{pmatrix} 3e^{t} - 2e^{2t} & -6e^{t} + 6e^{2t} \\ e^{t} - e^{2t} & -2e^{t} + 3e^{2t} \end{pmatrix}$$

$$= \begin{pmatrix} -3e^{t} + 2e^{2t} + 6e^{t} - 6e^{2t} & 6e^{t} - 6e^{2t} - 12e^{t} + 18e^{2t} \\ -3e^{t} + 2e^{2t} + 4e^{t} - 4e^{2t} & 6e^{t} - 6e^{2t} - 8e^{t} + 12e^{2t} \end{pmatrix}$$

$$= \begin{pmatrix} 3e^{t} - 4e^{2t} & -6e^{t} + 12e^{2t} \\ e^{t} - 2e^{2t} & -2e^{t} + 6e^{2t} \end{pmatrix} = \frac{d}{dt} e^{At} \checkmark$$

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3 $A = \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}$ $R = \begin{pmatrix} 1 & 1 \\ i & -i \end{pmatrix}$ $\Lambda = \begin{pmatrix} 1-i & 0 \\ 0 & 1+i \end{pmatrix}$

$$R^{-1} = \frac{1}{-i-i} \begin{pmatrix} -i & -1 \\ -i & 1 \end{pmatrix}$$

$$= \frac{1}{2} \begin{pmatrix} \frac{-i}{-i} & \frac{-1}{-i} \\ \frac{-i}{-i} & \frac{1}{-i} \end{pmatrix} = \frac{1}{2} \begin{pmatrix} 1 & -i \\ 1 & i \end{pmatrix}$$

a) $e^{At} = R e^{\Lambda t} R^{-1}$

$$= \begin{pmatrix} 1 & 1 \\ i & -i \end{pmatrix} \begin{pmatrix} e^{(1-i)t} & 0 \\ 0 & e^{(1+i)t} \end{pmatrix} \frac{1}{2} \begin{pmatrix} 1 & -i \\ 1 & i \end{pmatrix}$$

$$= \frac{e^t}{2} \begin{pmatrix} 1 & 1 \\ i & -i \end{pmatrix} \begin{pmatrix} e^{-it} & 0 \\ 0 & e^{it} \end{pmatrix} \begin{pmatrix} 1 & -i \\ 1 & i \end{pmatrix}$$

$$= \frac{e^t}{2} \begin{pmatrix} 1 & 1 \\ i & -i \end{pmatrix} \begin{pmatrix} e^{-it} & -ie^{-it} \\ e^{it} & ie^{it} \end{pmatrix}$$

$$= \frac{e^t}{2} \begin{pmatrix} e^{-it} + e^{it} & +i(-e^{-it} + e^{it}) \\ i(e^{-it} - e^{it}) & e^{-it} + e^{it} \end{pmatrix} \text{(over)}$$

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$$= e^t \begin{pmatrix} \frac{e^{it}+e^{-it}}{2} & \frac{e^{-it}-e^{it}}{2i} \\ \frac{-e^{-it}+e^{it}}{2i} & \frac{e^{it}+e^{-it}}{2} \end{pmatrix}$$

$$\boxed{= e^t \begin{pmatrix} \cos t & -\sin t \\ \sin t & \cos t \end{pmatrix}}$$

b) $$\frac{d}{dt} e^{At} = \begin{pmatrix} -e^t \sin t + e^t \cos t & -\cos t\, e^t - \sin t\, e^t \\ e^t \cos t + e^t \sin t & -e^t \sin t + e^t \cos t \end{pmatrix}$$

$$= e^t \begin{pmatrix} -\sin t + \cos t & -\cos t - \sin t \\ \cos t + \sin t & -\sin t + \cos t \end{pmatrix}$$

$$Ae^{At} = \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix} e^t \begin{pmatrix} \cos t & -\sin t \\ \sin t & \cos t \end{pmatrix}$$

$$= e^t \begin{pmatrix} \cos t - \sin t & -\sin t - \cos t \\ \cos t + \sin t & -\sin t + \cos t \end{pmatrix}$$

$$= \frac{d}{dt} e^{At}$$

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$$A = \begin{pmatrix} 3 & -1 & 1 \\ -2 & 4 & 3 \\ -1 & 1 & 5 \end{pmatrix} \quad R = \begin{array}{ccc} (\lambda=2) & (\lambda=4) & (\lambda=6) \\ \end{array} \begin{pmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \end{pmatrix}$$

I'll use elimination to compute R^{-1}

$$\left[\begin{array}{ccc|ccc} 1 & 1 & 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 & 0 & 1 \end{array}\right] \sim \left[\begin{array}{ccc|ccc} 1 & 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & -1 & 1 & -1 & 0 \\ 0 & 1 & 1 & 0 & 0 & 1 \end{array}\right]$$

$$\sim \left[\begin{array}{ccc|ccc} 1 & 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & -1 & 1 & -1 & 0 \\ 0 & 0 & 2 & -1 & 1 & 1 \end{array}\right] \sim \left[\begin{array}{ccc|ccc} 1 & 1 & 0 & 1 & 0 & 0 \\ 0 & 2 & 0 & 1 & -1 & 1 \\ 0 & 0 & 2 & -1 & 1 & 1 \end{array}\right]$$

$$\sim \left[\begin{array}{ccc|ccc} 2 & 0 & 0 & 1 & 1 & -1 \\ 0 & 2 & 0 & 1 & -1 & 1 \\ 0 & 0 & 2 & -1 & 1 & 1 \end{array}\right]$$

so $$R^{-1} = \frac{1}{2} \begin{pmatrix} 1 & 1 & -1 \\ 1 & -1 & 1 \\ -1 & 1 & 1 \end{pmatrix}$$

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Check R^{-1}:

$$\begin{pmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \end{pmatrix} \frac{1}{2} \begin{pmatrix} 1 & 1 & -1 \\ 1 & -1 & 1 \\ -1 & 1 & 1 \end{pmatrix}$$

$$= \frac{1}{2} \begin{pmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{pmatrix} = I \checkmark$$

a) $$e^{At} = R e^{\Lambda t} R^{-1} \qquad \Lambda = \begin{pmatrix} 2 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 6 \end{pmatrix}$$

$$= \begin{pmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \end{pmatrix} \begin{pmatrix} e^{2t} & 0 & 0 \\ 0 & e^{4t} & 0 \\ 0 & 0 & e^{6t} \end{pmatrix} \frac{1}{2} \begin{pmatrix} 1 & 1 & -1 \\ 1 & -1 & 1 \\ -1 & 1 & 1 \end{pmatrix}$$

$$= \frac{1}{2} \begin{pmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \end{pmatrix} \begin{pmatrix} e^{2t} & -e^{2t} & -e^{2t} \\ e^{4t} & -e^{4t} & e^{4t} \\ -e^{6t} & e^{6t} & e^{6t} \end{pmatrix}$$

$$= \frac{1}{2} \begin{pmatrix} e^{2t}+e^{4t} & e^{2t}-e^{4t} & -e^{2t}+e^{4t} \\ e^{2t}-e^{6t} & e^{2t}+e^{6t} & -e^{2t}+e^{6t} \\ e^{4t}-e^{6t} & -e^{4t}+e^{6t} & e^{4t}+e^{6t} \end{pmatrix}$$ Over

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b) $$\frac{d}{dt}e^{At} = \frac{1}{2}\begin{pmatrix} 2e^{2t}+4e^{4t} & 2e^{2t}-4e^{4t} & -2e^{2t}+4e^{4t} \\ 2e^{2t}-6e^{6t} & 2e^{2t}+6e^{6t} & -2e^{2t}+6e^{6t} \\ 4e^{4t}-6e^{6t} & -4e^{4t}+6e^{6t} & 4e^{4t}+6e^{6t} \end{pmatrix}$$

$$Ae^{At}$$

$$= \begin{pmatrix} 3 & -1 & 1 \\ -2 & 4 & 2 \\ -1 & 1 & 5 \end{pmatrix} \frac{1}{2}\begin{pmatrix} e^{2t}+e^{4t} & e^{2t}-e^{4t} & -e^{2t}+e^{4t} \\ e^{2t}-e^{6t} & e^{2t}+e^{6t} & -e^{2t}+e^{6t} \\ e^{4t}-e^{6t} & -e^{4t}+e^{6t} & e^{4t}+e^{6t} \end{pmatrix}$$

$$= \frac{1}{2}\begin{pmatrix} 3e^{2t}+3e^{4t} - e^{2t}+\cancel{e^{6t}} + e^{4t}-\cancel{e^{6t}} & 3e^{2t}-3e^{4t} - e^{2t}-\cancel{e^{6t}} - e^{4t}+\cancel{e^{6t}} & -3e^{2t}+3e^{4t} + e^{2t}-\cancel{e^{6t}} + e^{4t}+\cancel{e^{6t}} \\ -2e^{2t}-\cancel{2e^{4t}} + 4e^{2t}-4e^{6t} + \cancel{2e^{4t}}-2e^{6t} & -2e^{2t}+\cancel{2e^{4t}} + 4e^{2t}+4e^{6t} - \cancel{2e^{4t}}+2e^{6t} & 2e^{2t}-\cancel{2e^{4t}} - 4e^{2t}+4e^{6t} + \cancel{2e^{4t}}+2e^{6t} \\ -\cancel{e^{2t}}-e^{4t} + \cancel{e^{2t}}-e^{6t} + 5e^{4t}-5e^{6t} & -\cancel{e^{2t}}+e^{4t} + \cancel{e^{2t}}+e^{6t} - 5e^{4t}+5e^{6t} & \cancel{e^{2t}}-e^{4t} - \cancel{e^{2t}}+e^{6t} + 5e^{4t}+5e^{6t} \end{pmatrix}$$

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$$Ae^{At}$$

$$= \frac{1}{2}\begin{pmatrix} 2e^{2t}+4e^{4t} & 2e^{2t}-4e^{4t} & -2e^{2t}+4e^{4t} \\ 2e^{2t}-6e^{6t} & 2e^{2t}+6e^{6t} & -2e^{2t}+6e^{6t} \\ 4e^{4t}-6e^{6t} & -4e^{4t}+6e^{6t} & 4e^{4t}+6e^{6t} \end{pmatrix}$$

$$= \frac{d}{dt}e^{At}$$

5) $A = \begin{pmatrix} -1 & 6 \\ -1 & 4 \end{pmatrix}$, $R = \begin{array}{cc} (\lambda=1) & (\lambda=2) \\ \end{array}\begin{pmatrix} 3 & 2 \\ 1 & 1 \end{pmatrix}$ $R^{-1} = \begin{pmatrix} 1 & -2 \\ -1 & 3 \end{pmatrix}$

$$\log(I - At)$$

$$= R\log(I - \Lambda t)R^{-1}$$

But

$$\log(I - \Lambda t) = -\sum_{n=1}^{\infty}\frac{1}{n}(\Lambda t)^n$$

But $(\Lambda t)^n = \begin{pmatrix} (1t)^n & 0 \\ 0 & (2t)^n \end{pmatrix}$

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So

$$(*=)\ -\sum_{n=1}^{\infty}\frac{1}{n}(Λt)^n = \begin{pmatrix} -\sum_{n=1}^{\infty}\frac{1}{n}t^n & 0 \\ 0 & -\sum_{n=1}^{\infty}\frac{1}{n}(2t)^n \end{pmatrix}$$

But if and only if $|t|<1$

$$-\sum_{n=1}^{\infty}\frac{1}{n}t^n = \log(1-t)$$

and if and only if $|2t|<1$

$$-\sum_{n=1}^{\infty}\frac{1}{n}(2t)^n = \log(1-2t)$$

So if $|2t|<1 \Rightarrow |t|<1/2$

$$* = \begin{pmatrix} \log(1-t) & 0 \\ 0 & \log(1-2t) \end{pmatrix}$$

So

$$\log(I-At) = \begin{pmatrix} 3 & 2 \\ 1 & 1 \end{pmatrix}\begin{pmatrix} \log(1-t) & 0 \\ 0 & \log(1-2t) \end{pmatrix}\begin{pmatrix} 1 & -2 \\ -1 & 3 \end{pmatrix}$$

$$= \begin{pmatrix} 3 & 2 \\ 1 & 1 \end{pmatrix}\begin{pmatrix} \log(1-t) & -2\log(1-t) \\ -\log(1-2t) & 3\log(1-2t) \end{pmatrix} \to \text{(over)}$$

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$$= \begin{pmatrix} 3\log(1-t) - 2\log(1-2t) & -6\log(1-t) + 6\log(1-2t) \\ \log(1-t) - \log(1-2t) & -2\log(1-t) + 3\log(1-2t) \end{pmatrix}$$

$$= \log(I - At)$$

b) $\frac{d}{dt}\log(I-At)$

$$= \begin{pmatrix} \frac{-3}{1-t} + \frac{4}{1-2t} & \frac{6}{1-t} - \frac{12}{1-2t} \\ \frac{-1}{1-t} + \frac{2}{1-2t} & \frac{2}{1-t} - \frac{6}{1-2t} \end{pmatrix}$$

Factor out

$$\frac{1}{(1-t)(1-2t)} \begin{pmatrix} -3(1-2t) + 4(1-t) & 6(1-2t) - 12(1-t) \\ -(1-2t) + 2(1-t) & 2(1-2t) - 6(1-t) \end{pmatrix}$$

$$= \frac{1}{(1-t)(1-2t)} \begin{pmatrix} 1+2t & -6 \\ 1 & -4+2t \end{pmatrix}$$

$$\| $$

$$\frac{d}{dt}\log(I-At)$$

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$$(I - At) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} - t \begin{pmatrix} -1 & 6 \\ -1 & 4 \end{pmatrix}$$

$$= \begin{pmatrix} 1+t & -6t \\ t & 1-4t \end{pmatrix}$$

so $$(I - At)^{-1} = \frac{1}{(1+t)(1-4t)+6t^2} \begin{pmatrix} 1-4t & 6t \\ -t & 1+t \end{pmatrix}$$

use 2x2 Cramer's trick

$$= \frac{1}{1-3t+2t^2} \begin{pmatrix} 1-4t & 6t \\ -t & 1+t \end{pmatrix}$$

Factor

$$= \frac{1}{(1-t)(1-2t)} \begin{pmatrix} 1-4t & 6t \\ -t & 1+t \end{pmatrix}$$

so $-A(I - At)^{-1}$

$$= \begin{pmatrix} 1 & -6 \\ 1 & -4 \end{pmatrix} \frac{1}{(1-t)(1-2t)} \begin{pmatrix} 1-4t & 6t \\ -t & 1+t \end{pmatrix}$$

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$$= \frac{1}{(1-t)(1-2t)} \begin{pmatrix} 1-4t+6t & 6t-6(1+t) \\ 1-4t+4t & 6t-4(1+t) \end{pmatrix}$$

$$= \frac{1}{(1-t)(1-2t)} \begin{pmatrix} 1+2t & -6 \\ 1 & -4+2t \end{pmatrix} \checkmark$$

$$= \frac{d}{dt} \log(I - At)$$ (see bottom of page 11)

6 $A = \begin{pmatrix} -5 & 18 \\ -3 & 10 \end{pmatrix}$ $\lambda = 1$ $r = \begin{pmatrix} 3 \\ 1 \end{pmatrix}$

$\lambda = 4$ $r = \begin{pmatrix} 2 \\ 1 \end{pmatrix}$

$R = \begin{pmatrix} 3 & 2 \\ 1 & 1 \end{pmatrix}$ $R^{-1} = \begin{pmatrix} 1 & -2 \\ -1 & 3 \end{pmatrix}$

$$A = R \Lambda R^{-1} = \begin{pmatrix} 3 & 2 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 4 \end{pmatrix} \begin{pmatrix} 1 & -2 \\ -1 & 3 \end{pmatrix}$$

But $\begin{pmatrix} 1 & 0 \\ 0 & 4 \end{pmatrix} = \begin{pmatrix} \sqrt{1} & 0 \\ 0 & \sqrt{4} \end{pmatrix} \begin{pmatrix} \sqrt{1} & 0 \\ 0 & \sqrt{4} \end{pmatrix} = \sqrt{\Lambda}\sqrt{\Lambda}$

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So

$$A = R\sqrt{\Lambda}\,\sqrt{\Lambda}\,R^{-1}$$

$$= R\sqrt{\Lambda}\,R^{-1}\,R\sqrt{\Lambda}\,R^{-1} = \left(R\sqrt{\Lambda}\,R^{-1}\right)^2$$

So

$$R\sqrt{\Lambda}\,R^{-1} = \begin{pmatrix} 3 & 2 \\ 1 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix}\begin{pmatrix} 1 & -2 \\ -1 & 3 \end{pmatrix}$$

$$= \begin{pmatrix} 3 & 2 \\ 1 & 1 \end{pmatrix}\begin{pmatrix} 1 & -2 \\ -2 & 6 \end{pmatrix} = \boxed{\begin{pmatrix} -1 & 6 \\ -1 & 4 \end{pmatrix} = \sqrt{A}}$$

check:

$$\begin{pmatrix} -1 & 6 \\ -1 & 4 \end{pmatrix}\begin{pmatrix} -1 & 6 \\ -1 & 4 \end{pmatrix}$$

$$= \begin{pmatrix} -5 & 18 \\ -3 & 10 \end{pmatrix} \checkmark$$