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Math 2331 HW 7

1a) $p(\lambda) = \det(A - \lambda I) = \det\begin{pmatrix} 1-\lambda & 2 \\ 2 & 1-\lambda \end{pmatrix}$

$= (1-\lambda)(1-\lambda) - 4 = \boxed{\lambda^2 - 2\lambda - 3}$

$p(\lambda) = 0 \Rightarrow \lambda = \frac{2 \pm \sqrt{4+12}}{2} = 1 \pm 2$

So $\boxed{\lambda = -1 \text{ \& } \lambda = 3}$

b) $p(\lambda) = \det\begin{pmatrix} 1-\lambda & 1 \\ 0 & 1-\lambda \end{pmatrix} = \boxed{(1-\lambda)^2}$

So here we have only one eigenvalue, $\boxed{\lambda = 1}$ and it has multip 2

c) $p(\lambda) = \det\begin{pmatrix} -1-\lambda & 6 \\ -1 & 4-\lambda \end{pmatrix}$

$= (\lambda-4)(\lambda+1) + 6 = \boxed{\lambda^2 - 3\lambda + 2}$

$p(\lambda) = 0 \Rightarrow \lambda = \frac{3 \pm \sqrt{9 - 4 \cdot 2}}{2}$

So $\boxed{\lambda = 1 \text{ \& } \lambda = 2}$

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d) $p(\lambda)=\det\begin{pmatrix}1-\lambda & -1\\ 1 & 1-\lambda\end{pmatrix}=(1-\lambda)^2+1$

$=\lambda^2-2\lambda+1+1=\boxed{\lambda^2-2\lambda+2}$

$p(\lambda)=0 \Rightarrow \lambda=\frac{2\pm\sqrt{4-4\cdot 2}}{2}$

$=\frac{2\pm\sqrt{-4}}{2}=1\pm i \quad \boxed{\lambda=1-i \ \&\ \lambda=1+i}$

2a)

$p(\lambda)=\det\begin{pmatrix}1-\lambda & 2 & 1\\ 2 & 1-\lambda & 2\\ 0 & 0 & 3-\lambda\end{pmatrix}$

$=(3-\lambda)\det\begin{pmatrix}1-\lambda & 2\\ 2 & 1-\lambda\end{pmatrix}$

$=(3-\lambda)\left[(1-\lambda)^2-4\right]$

$=(3-\lambda)\left[\lambda^2-2\lambda-3\right]$

$=\boxed{-(\lambda-3)(\lambda+1)(\lambda-3)=-(\lambda+1)(\lambda-3)^2}$

$p(\lambda)=0 \Rightarrow \boxed{\lambda=-1 \ \&\ \lambda=3 \text{ mult } 2,}$

2b) $$p(\lambda) = \det\begin{pmatrix} 3-\lambda & -1 & 1 \\ -2 & 4-\lambda & 2 \\ -1 & 1 & 5-\lambda \end{pmatrix}$$

$$= \left[(3-\lambda)(4-\lambda)(5-\lambda) + 2 - 2\right]$$

$$- \left[-(4-\lambda) + 2(3-\lambda) + 2(5-\lambda)\right]$$

$$= \left[(3-\lambda)(4-\lambda)(5-\lambda)\right]$$

$$- \left[12 - 3\lambda\right]$$

$$= (3-\lambda)(4-\lambda)(5-\lambda) + 3(\lambda - 4)$$

$$= -(\lambda-4)\left[(\lambda-3)(\lambda-5) - 3\right]$$

$$= -(\lambda-4)\left[\lambda^2 - 8\lambda + 12\right]$$

$$= \boxed{-(\lambda-4)(\lambda-2)(\lambda-6)}$$

$$p(\lambda) = 0 \Rightarrow \boxed{\lambda = 2, \lambda = 4, \lambda = 6}$$

cofactor

3a)

$$p(\lambda) = \det\begin{pmatrix} 3-\lambda & 0 & -1 & 1 \\ 1 & 8-\lambda & 2 & 3 \\ -2 & 0 & 4-\lambda & 2 \\ -1 & 0 & 1 & 5-\lambda \end{pmatrix}$$

$$= (-1)^{2+2}(8-\lambda)\det\begin{pmatrix} 3-\lambda & -1 & 1 \\ -2 & 4-\lambda & 2 \\ -1 & 1 & 5-\lambda \end{pmatrix}$$

$$= (8-\lambda)\Big[\big[(3-\lambda)(4-\lambda)(5-\lambda) + 2 - 2\big] \big[-(4-\lambda) + 2(3-\lambda) + 2(5-\lambda)\big]\Big]$$

From 2b

$$= (8-\lambda)\big[-(\lambda-4)(\lambda-2)(\lambda-6)\big]$$

$$= \boxed{(\lambda-2)(\lambda-4)(\lambda-6)(\lambda-8)}$$

$$p(\lambda) = 0 \Rightarrow \boxed{\lambda = 2, \lambda = 4, \lambda = 6, \lambda = 8}$$

cofactor on row 1

3b)

$$P(\lambda) = \det\begin{pmatrix} 1-\lambda & 2 & 0 & 0 \\ 2 & 1-\lambda & 0 & 0 \\ 0 & 0 & 1-\lambda & 2 \\ 0 & 0 & 2 & 1-\lambda \end{pmatrix}$$

$$= (-1)^{1+1}(1-\lambda)\det\begin{pmatrix} 1-\lambda & 0 & 0 \\ 0 & 1-\lambda & 2 \\ 0 & 2 & 1-\lambda \end{pmatrix}$$

$$+ (-1)^{1+2}\, 2 \det\begin{pmatrix} 2 & 0 & 0 \\ 0 & 1-\lambda & 2 \\ 0 & 2 & 1-\lambda \end{pmatrix}$$

$$= (1-\lambda)\left[(1-\lambda)\left[(1-\lambda)^2 - 4\right]\right]$$

$$- 2\left[2\left[(1-\lambda)^2 - 4\right]\right]$$

$$= \left((1-\lambda)^2 - 4\right)\left((1-\lambda)^2 - 4\right)$$

$$= (\lambda+1)(\lambda-3)(\lambda+1)(\lambda-3)$$

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$$= \boxed{(\lambda+1)^2(\lambda-3)^2}$$

$$\boxed{\begin{array}{l} \lambda=-1 \text{ with mult } 2 \\ \lambda=3 \text{ with mult } 2 \end{array}}$$

4) Find eigenvectors for

a) $\begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix}, \lambda=-1, 3$

$$\lambda=-1: \begin{bmatrix} 1-(-1) & 2 \\ 2 & 1-(-1) \end{bmatrix} \sim \begin{bmatrix} 2 & 2 \\ 2 & 2 \end{bmatrix}$$

$$\sim \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix} \quad \boxed{r = \begin{pmatrix} 1 \\ -1 \end{pmatrix}}$$

$$\lambda=3: \begin{bmatrix} 1-3 & 2 \\ 2 & 1-3 \end{bmatrix} \sim \begin{bmatrix} -2 & 2 \\ 2 & -2 \end{bmatrix}$$

$$\sim \begin{bmatrix} 1 & -1 \\ 0 & 0 \end{bmatrix} \quad \boxed{r = \begin{pmatrix} 1 \\ 1 \end{pmatrix}}$$

b) $\begin{pmatrix}1 & 1\\ 0 & 1\end{pmatrix}$, $\lambda = 1$

$\lambda=1$: $\begin{bmatrix}1-1 & 1\\ 0 & 1-1\end{bmatrix} \sim \begin{bmatrix}0 & 1\\ 0 & 0\end{bmatrix}$

$\boxed{r = \begin{pmatrix}1\\ 0\end{pmatrix}}$ and there's only one.

c) $\begin{pmatrix}-1 & 6\\ -1 & 4\end{pmatrix}$ $\lambda = 1, 2$

$\lambda=1$: $\begin{bmatrix}-1-1 & 6\\ -1 & 4-1\end{bmatrix} \sim \begin{bmatrix}-2 & 6\\ -1 & 3\end{bmatrix}$

$\sim \begin{bmatrix}1 & -3\\ 0 & 0\end{bmatrix}$ $\boxed{r = \begin{pmatrix}3\\ 1\end{pmatrix}}$

$\lambda=2$: $\begin{bmatrix}-1-2 & 6\\ -1 & 4-2\end{bmatrix} \sim \begin{bmatrix}-3 & 6\\ -1 & 2\end{bmatrix}$

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$\sim \begin{bmatrix} 1 & -2 \\ 0 & 0 \end{bmatrix}$ $\boxed{r = \begin{pmatrix} 2 \\ 1 \end{pmatrix}}$

d) $\begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}$ $\lambda = 1-i, 1+i$

$\lambda = 1-i$ $\begin{bmatrix} 1-(1-i) & -1 \\ 1 & 1-(1-i) \end{bmatrix} \sim \begin{bmatrix} i & -1 \\ 1 & i \end{bmatrix}$

$\sim \begin{bmatrix} +1 & +i \\ 1 & i \end{bmatrix} \sim \begin{bmatrix} 1 & i \\ 0 & 0 \end{bmatrix}$ $\boxed{r = \begin{pmatrix} 1 \\ i \end{pmatrix}}$

$\lambda = 1+i$ $\begin{bmatrix} 1-(1+i) & -1 \\ 1 & 1-(1+i) \end{bmatrix} \sim \begin{bmatrix} -i & -1 \\ 1 & -i \end{bmatrix}$

$\sim \begin{bmatrix} 1 & -i \\ 1 & -i \end{bmatrix} \sim \begin{bmatrix} 1 & -i \\ 0 & 0 \end{bmatrix}$ $\boxed{r = \begin{pmatrix} 1 \\ -i \end{pmatrix}}$

5) Find eigenvectors

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(a) $\begin{pmatrix} 1 & 2 & 1 \\ 2 & 1 & 2 \\ 0 & 0 & 3 \end{pmatrix}$, $\lambda = -1$, $\lambda = 3$

$$\lambda = -1: \begin{bmatrix} 1-(-1) & 2 & 1 \\ 2 & 1-(-1) & 2 \\ 0 & 0 & 3-(-1) \end{bmatrix} \sim \begin{bmatrix} 2 & 2 & 1 \\ 2 & 2 & 2 \\ 0 & 0 & 4 \end{bmatrix}$$

$$\sim \begin{bmatrix} 2 & 2 & 1 \\ 0 & 0 & 1 \\ 0 & 0 & 1 \end{bmatrix} \sim \begin{bmatrix} 2 & 2 & 1 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

$x_3 = 0$ $\therefore x_1 + x_2 = 0$ $\boxed{r = \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix}}$

$$\lambda = 3: \begin{bmatrix} 1-3 & 2 & 1 \\ 2 & 1-3 & 2 \\ 0 & 0 & 0 \end{bmatrix}$$

$$\sim \begin{bmatrix} -2 & 2 & 1 \\ 2 & -2 & 2 \\ 0 & 0 & 0 \end{bmatrix} \sim \begin{bmatrix} 2 & -2 & -1 \\ 0 & 0 & 3 \\ 0 & 0 & 0 \end{bmatrix}$$

$$\sim \begin{bmatrix} 1 & -1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$ $x_3 = 0$ $x_1 - x_2 = 0$ $\boxed{r = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}}$

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and This has only one eigenvector.

b) $\begin{pmatrix} 3 & -1 & 1 \\ -2 & 4 & 2 \\ -1 & 1 & 5 \end{pmatrix}$ $\lambda = 2, \lambda = 4, \lambda = 6$

$\lambda = 2$: $\begin{bmatrix} 3-2 & -1 & 1 \\ -2 & 4-2 & 2 \\ -1 & 1 & 5-2 \end{bmatrix} \sim \begin{bmatrix} 1 & -1 & 1 \\ -2 & 2 & 2 \\ -1 & 1 & 3 \end{bmatrix}$

$\sim \begin{bmatrix} 1 & -1 & 1 \\ -1 & 1 & 1 \\ -1 & 1 & 3 \end{bmatrix} \sim \begin{bmatrix} 1 & -1 & 1 \\ 0 & 0 & 2 \\ 0 & 0 & 4 \end{bmatrix}$

$\sim \begin{bmatrix} 1 & -1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$ $x_3 = 0$, $x_1 = x_2$ $\boxed{r = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}}$

$\lambda = 4$ $\begin{bmatrix} 3-4 & -1 & 1 \\ -2 & 4-4 & 2 \\ -1 & 1 & 5-4 \end{bmatrix} \sim \begin{bmatrix} -1 & -1 & 1 \\ -2 & 0 & 2 \\ -1 & 1 & 1 \end{bmatrix}$

$\sim \begin{bmatrix} 1 & 0 & -1 \\ -1 & -1 & 1 \\ -1 & 1 & 1 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & -1 \\ 0 & -1 & 0 \\ 0 & 1 & 0 \end{bmatrix}$ (over)

$$\sim\begin{bmatrix}1 & 0 & -1\\ 0 & 1 & 0\\ 0 & 0 & 0\end{bmatrix}$$

$x_2=0 \quad x_1-x_3=0$

$$\boxed{r=\begin{pmatrix}1\\0\\1\end{pmatrix}}$$

$\lambda=6$:

$$\begin{bmatrix}3-6 & -1 & 1\\ -2 & 4-6 & 2\\ -1 & 1 & 5-6\end{bmatrix}\sim\begin{bmatrix}-3 & -1 & 1\\ -2 & -2 & 2\\ -1 & 1 & -1\end{bmatrix}$$

$$\sim\begin{bmatrix}3 & 1 & -1\\ 1 & 1 & -1\\ 1 & -1 & 1\end{bmatrix}\sim\begin{bmatrix}\overset{3}{1} & \overset{3}{1} & \overset{-3}{-1}\\ 3 & 1 & -1\\ 1 & -1 & 1\end{bmatrix}$$

$$\sim\begin{bmatrix}1 & 1 & -1\\ 0 & -2 & 2\\ 0 & -2 & 2\end{bmatrix}\sim\begin{bmatrix}1 & 1 & -1\\ 0 & 1 & -1\\ 0 & 0 & 0\end{bmatrix}$$

$$\sim\begin{bmatrix}1 & 0 & 0\\ 0 & 1 & -1\\ 0 & 0 & 0\end{bmatrix}$$

$x_1=0 \quad x_2-x_3=0$

$$\boxed{r=\begin{pmatrix}0\\1\\1\end{pmatrix}}$$

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6) Find eigenvectors

a) $\begin{pmatrix} 3 & 0 & -1 & 1 \\ 1 & 8 & 2 & 3 \\ -2 & 0 & 4 & 2 \\ -1 & 0 & 1 & 5 \end{pmatrix}$ $\lambda=2$ $\lambda=4$ $\lambda=6$ $\lambda=8$

$\lambda=2$; $\begin{bmatrix} 1 & 0 & -1 & 1 \\ 1 & 6 & 2 & 3 \\ -2 & 0 & 2 & 2 \\ -1 & 0 & 1 & 3 \end{bmatrix}$

$\sim \begin{bmatrix} 1 & 0 & -1 & 1 \\ 0 & 6 & 3 & 2 \\ 0 & 0 & 0 & 2 \\ 0 & 0 & 0 & 4 \end{bmatrix}$

$\sim \begin{bmatrix} 1 & 0 & -1 & 0 \\ 0 & 6 & 3 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & -1 & 0 \\ 0 & 2 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$

$x_4=0$ $2x_2+x_3=0$

$x_1-x_3=0$

$$\boxed{r=\begin{pmatrix} 2 \\ -1 \\ 2 \\ 0 \end{pmatrix}}$$

$$\lambda=4: \begin{bmatrix} -1 & 0 & -1 & 1 \\ 1 & 4 & 2 & 3 \\ -2 & 0 & 0 & 2 \\ -1 & 0 & 1 & 1 \end{bmatrix}$$

$$\sim \begin{bmatrix} 1 & 0 & 1 & -1 \\ 1 & 4 & 2 & 3 \\ 1 & 0 & 0 & -1 \\ 1 & 0 & -1 & -1 \end{bmatrix}$$

$$\sim \begin{bmatrix} 1 & 0 & 1 & -1 \\ 0 & 4 & 1 & 4 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & -2 & 0 \end{bmatrix}$$

$$\sim \begin{bmatrix} 1 & 0 & 0 & -1 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$X_3 = 0$

$X_2 + X_4 = 0$

$X_1 - X_4 = 0$

$$\boxed{r = \begin{pmatrix} 1 \\ -1 \\ 0 \\ 1 \end{pmatrix}}$$

$$\lambda=6\quad\begin{bmatrix}-3 & 0 & -1 & 1\\ 1 & 2 & 2 & 3\\ -2 & 0 & -2 & 2\\ -1 & 0 & 1 & -1\end{bmatrix}$$

$$\sim\begin{array}{c}\begin{matrix}3 & 0 & -3 & 3\end{matrix}\\ \begin{bmatrix}1 & 0 & -1 & 1\\ 1 & 2 & 2 & 3\\ -1 & 0 & 1 & -1\\ 3 & 0 & 1 & -1\end{bmatrix}\end{array}$$

$$\sim\begin{bmatrix}1 & 0 & -1 & 1\\ 0 & 2 & 3 & 2\\ 0 & 0 & 2 & -2\\ 0 & 0 & 4 & -4\end{bmatrix}$$

$$\sim\begin{bmatrix}1 & 0 & -1 & 1\\ 0 & 2 & 3 & 2\\ 0 & 0 & 1 & -1\\ 0 & 0 & 0 & 0\end{bmatrix}\qquad 2-1\,3)$$

$$\sim\begin{bmatrix}1 & 0 & 0 & 0\\ 0 & 2 & 0 & 5\\ 0 & 0 & 1 & -1\\ 0 & 0 & 0 & 0\end{bmatrix}$$

$x_4=2$

$x_3=2$

$2x_2+5,2$

$x_1=0$

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$$\boxed{r = \begin{pmatrix} 0 \\ -5 \\ 2 \\ 2 \end{pmatrix}}$$

$$\lambda = 8: \begin{bmatrix} -5 & 0 & -1 & 1 \\ 1 & 0 & 2 & 3 \\ -2 & 0 & -4 & 2 \\ -1 & 0 & 1 & -3 \end{bmatrix} \begin{array}{c} \\ \checkmark \\ \\ \checkmark \end{array}$$

$$\sim \begin{bmatrix} \overset{5}{1} & \overset{0}{0} & \overset{-5}{-1} & \overset{5}{3} \\ 1 & 0 & 2 & 3 \\ 1 & 0 & 2 & -1 \\ -5 & 0 & -1 & 1 \end{bmatrix}$$

$$\sim \begin{bmatrix} 1 & 0 & -1 & 3 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 3 & -4 \\ 0 & 0 & -6 & 16 \end{bmatrix}$$

$$\sim \begin{bmatrix} 1 & 0 & 0 & 3 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -4 \\ 0 & 0 & 0 & 16 \end{bmatrix}$$

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$$\sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$x_4 = 0$
$x_3 = 0$
$x_2 = 1$
$x_1 = 0$

$$\boxed{r = \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix}}$$

b) $$\begin{pmatrix} 1 & 2 & 0 & 0 \\ 2 & 1 & 0 & 0 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 2 & 1 \end{pmatrix}$$ $\lambda = -1$ $\lambda = 3$

$\lambda = -1$

NOTE: This eval has mult 2

$$\begin{bmatrix} 1-(-1) & 2 & 0 & 0 \\ 2 & 1-(-1) & 0 & 0 \\ 0 & 0 & 1-(-1) & 2 \\ 0 & 0 & 2 & 1-(-1) \end{bmatrix}$$

$$\sim \begin{bmatrix} 2 & 2 & 0 & 0 \\ 2 & 2 & 0 & 0 \\ 0 & 0 & 2 & 2 \\ 0 & 0 & 2 & 2 \end{bmatrix}$$

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$$\begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\sim \begin{array}{c} \begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \\ \begin{array}{cccc} \quad & \uparrow & \quad & \uparrow \\ & \text{free} & & \text{free} \end{array} \end{array}$$

Let $x_2 = \alpha$ $\quad x_4 = \beta$

$x_1 = -\alpha$ $\quad x_3 = -\beta$

$$r = \begin{pmatrix} -\alpha \\ \alpha \\ -\beta \\ \beta \end{pmatrix} = -\alpha \begin{pmatrix} 1 \\ -1 \\ 0 \\ 0 \end{pmatrix} - \beta \begin{pmatrix} 0 \\ 0 \\ 1 \\ -1 \end{pmatrix}$$

So $\lambda = -1$ has $\underline{\underline{2}}$ independant eigenvectors

$$\boxed{r = \begin{pmatrix} 1 \\ -1 \\ 0 \\ 0 \end{pmatrix}} \;\&\; \boxed{r = \begin{pmatrix} 0 \\ 0 \\ 1 \\ -1 \end{pmatrix}}$$

$\lambda=3$

This eval also has multiplicity 2

$$\begin{bmatrix} -2 & 2 & 0 & 0 \\ 2 & -2 & 0 & 0 \\ 0 & 0 & -2 & 2 \\ 0 & 0 & 2 & -2 \end{bmatrix}$$

$$\sim \begin{bmatrix} 1 & -1 & 0 & 0 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$\uparrow$ free $\qquad \uparrow$ free

$x_2=\alpha \qquad x_4=\beta$

$x_1=\alpha \qquad x_3=\beta$

$$r=\begin{pmatrix}\alpha\\ \alpha\\ \beta\\ \beta\end{pmatrix}=\alpha\begin{pmatrix}1\\1\\0\\0\end{pmatrix}+\beta\begin{pmatrix}0\\0\\1\\1\end{pmatrix}$$

So here too, $\lambda=\underline{3}$ has 2 independent eigenvectors

$$\boxed{r=\begin{pmatrix}1\\1\\0\\0\end{pmatrix}} \;\&\; \boxed{r=\begin{pmatrix}0\\0\\1\\1\end{pmatrix}}$$