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Linear Alg (Part III)

1) Suppose $AB = I$.

a) $BX = x_1 \vec{b}_1 + \cdots + x_m \vec{b}_m = \vec{0}$

$\vec{0} = AB\vec{X} = A(x_1 \vec{b}_1 + \cdots x_m \vec{b}_m)$

$= ABX = IX$

$= x_1 \vec{e}_1 + \cdots + x_m \vec{e}_m$

But $\{\vec{e}_1 \cdots \vec{e}_m\}$ is clearly independent.

That is $\vec{0} = x_1 \vec{e}_1 + \cdots + x_m e_m$

$\Rightarrow x_1 = \cdots = x_m = 0$

$\therefore x_1 \vec{b}_1 + \cdots + x_m \vec{b}_m = \vec{0} \Rightarrow x_1 = \cdots = x_m = 0$

and this says $\{b_1 \cdots b_m\}$ is independent.

b) Since the columns of B are independent we know B is invertible.

c) Suppose $A\vec{X} = \vec{0}$.

Let $\vec{X} = B\vec{y}$ (i.e. $\vec{y} = B^{-1}\vec{X}$)

$\Rightarrow \vec{0} = A\vec{X} = AB\vec{y} = \vec{y}$

Since $\vec{y} = \vec{0}$ then $\Rightarrow \vec{x} = B\vec{y} = \vec{0}$

That is $A\vec{x} = \vec{0} \Rightarrow \vec{x} = \vec{0}$.

d) $A\vec{x} = x_1 \vec{a}_1 + \cdots + x_m \vec{a}_m = 0$

$\Rightarrow x_1 = \cdots = x_m = 0$ from (c).

But this says $\{\vec{a}_1, \cdots, \vec{a}_m\}$ is independent.

2a) $\left[\begin{array}{cc|cc} 1 & 2 & 1 & 0 \\ 2 & 1 & 0 & 1 \end{array}\right]$

$\sim \left[\begin{array}{cc|cc} 1 & 2 & 1 & 0 \\ 0 & -3 & -2 & 1 \end{array}\right] \sim \left[\begin{array}{cc|cc} 1 & 2 & 1 & 0 \\ 0 & 1 & 2/3 & -1/3 \end{array}\right]$

$\sim \left[\begin{array}{cc|cc} 1 & 0 & -1/3 & 2/3 \\ 0 & 1 & 2/3 & -1/3 \end{array}\right] \Rightarrow \boxed{A^{-1} = \begin{pmatrix} -1/3 & 2/3 \\ 2/3 & -1/3 \end{pmatrix}}$

b) $\left[\begin{array}{cc|cc} 1 & 2 & 1 & 0 \\ 3 & 1 & 0 & 1 \end{array}\right] \sim \left[\begin{array}{cc|cc} 1 & 2 & 1 & 0 \\ 0 & -5 & -3 & 1 \end{array}\right]$

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(3)

$$\sim \left[\begin{array}{cc|cc} 1 & 2 & 1 & 0 \\ 0 & 1 & 3/5 & -1/5 \end{array}\right]$$

$$\sim \left[\begin{array}{cc|cc} 1 & 0 & -1/5 & 2/5 \\ 0 & 1 & 3/5 & -1/5 \end{array}\right] \Rightarrow \boxed{A^{-1} = \begin{pmatrix} -1/5 & 2/5 \\ 3/5 & -1/5 \end{pmatrix}}$$

c) $$\left[\begin{array}{cc|cc} 1 & 2 & 1 & 0 \\ 2 & 4 & 0 & 1 \end{array}\right] \sim \left[\begin{array}{cc|cc} 1 & 2 & 1 & 0 \\ 0 & 0 & -2 & 1 \end{array}\right]$$

NOT INVERTIBLE.

d) $$\left[\begin{array}{cc|cc} 4 & 3 & 1 & 0 \\ 2 & 1 & 0 & 1 \end{array}\right] \sim \left[\begin{array}{cc|cc} 2 & 1 & 0 & 1 \\ 0 & 1 & 1 & -2 \end{array}\right]$$

$$\sim \left[\begin{array}{cc|cc} 2 & 0 & -1 & 3 \\ 0 & 1 & 1 & -2 \end{array}\right]$$

$$\sim \left[\begin{array}{cc|cc} 1 & 0 & -1/2 & 3/2 \\ 0 & 1 & 1 & -2 \end{array}\right] \quad \boxed{A^{-1} = \begin{pmatrix} -1/2 & 3/2 \\ 1 & -2 \end{pmatrix}}$$

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3a) $$\left[\begin{array}{ccc|ccc} 1 & 1 & 1 & 1 & 0 & 0 \\ 3 & 4 & 1 & 0 & 1 & 0 \\ 2 & 4 & 1 & 0 & 0 & 1 \end{array}\right]$$

$$\sim \left[\begin{array}{ccc|ccc} 1 & 1 & 1 & 1 & 0 & 0 \\ 0 & 1 & -2 & -3 & 1 & 0 \\ 0 & 2 & -1 & -2 & 0 & 1 \end{array}\right]$$

$$\sim \left[\begin{array}{ccc|ccc} 1 & 1 & 1 & 1 & 0 & 0 \\ 0 & 1 & -2 & -3 & 1 & 0 \\ 0 & 0 & 3 & 4 & -2 & 1 \end{array}\right]$$

$$\sim \left[\begin{array}{ccc|ccc} 1 & 1 & 1 & 1 & 0 & 0 \\ 0 & 1 & -2 & -3 & 1 & 0 \\ 0 & 0 & 1 & 4/3 & -2/3 & 1/3 \end{array}\right]$$

$$\sim \left[\begin{array}{ccc|ccc} 1 & 1 & 0 & -1/3 & 2/3 & -1/3 \\ 0 & 1 & 0 & -1/3 & -1/3 & 2/3 \\ 0 & 0 & 1 & 4/3 & -2/3 & 1/3 \end{array}\right]$$

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$$\sim \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 0 & 3/3 & -3/3 \\ 0 & 1 & 0 & -1/3 & -1/3 & 2/3 \\ 0 & 0 & 1 & 4/3 & -2/3 & 1/3 \end{array}\right]$$

So $$\boxed{A^{-1} = \begin{pmatrix} 0 & 1 & -1 \\ -1/3 & -1/3 & 2/3 \\ 4/3 & -2/3 & 1/3 \end{pmatrix}}$$

b) $$\left[\begin{array}{ccc|ccc} 1 & 2 & 3 & 1 & 0 & 0 \\ -2 & -2 & -5 & 0 & 1 & 0 \\ 2 & 2 & 8 & 0 & 0 & 1 \end{array}\right]$$

$$\sim \left[\begin{array}{ccc|ccc} 1 & 2 & 3 & 1 & 0 & 0 \\ 0 & 2 & 1 & 2 & 1 & 0 \\ 0 & 2 & -2 & 2 & 0 & -1 \end{array}\right]$$

$$\sim \left[\begin{array}{ccc|ccc} 1 & 2 & 3 & 1 & 0 & 0 \\ 0 & 2 & 1 & 2 & 1 & 0 \\ 0 & 0 & 3 & 0 & 1 & 1 \end{array}\right]$$

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$$\sim \left[\begin{array}{ccc|ccc} 1 & 2 & 3 & 1 & 0 & 0 \\ 0 & 2 & 1 & 2 & 1 & 0 \\ 0 & 0 & 1 & 0 & 1/3 & 1/3 \end{array}\right]$$

$$\sim \left[\begin{array}{ccc|ccc} 1 & 2 & 0 & 1 & -3/3 & -3/3 \\ 0 & 2 & 0 & 2 & 2/3 & -1/3 \\ 0 & 0 & 1 & 0 & 1/3 & 1/3 \end{array}\right]$$

$$\sim \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & -1 & -5/3 & -2/3 \\ 0 & 2 & 0 & 2 & 2/3 & -1/3 \\ 0 & 0 & 1 & 0 & 1/3 & 1/3 \end{array}\right]$$

so $\boxed{A^{-1} = \begin{pmatrix} -1 & -5/3 & -2/3 \\ 1 & 1/3 & -1/6 \\ 0 & 1/3 & 1/3 \end{pmatrix}}$

4a) $\left[\begin{array}{cccc|cccc} 1 & 1 & 1 & 1 & 1 & 0 & 0 & 0 \\ 1 & 2 & 1 & 1 & 0 & 1 & 0 & 0 \\ 1 & 2 & 1 & 1 & 0 & 0 & 1 & 0 \\ 1 & 2 & 2 & 3 & 0 & 0 & 0 & 1 \end{array}\right]$

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$$\left[\begin{array}{cccc|cccc} 1 & 1 & 1 & 1 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & -1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & -1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 2 & -1 & 0 & 0 & 1 \end{array}\right]$$

$$\sim \left[\begin{array}{cccc|cccc} 1 & 1 & 1 & 1 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & -1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 & 1 & 0 \\ 0 & 0 & 1 & 2 & 0 & -1 & 0 & 1 \end{array}\right]$$

$$\sim \left[\begin{array}{cccc|cccc} 1 & 1 & 1 & 1 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & -1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 2 & 0 & -1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & -1 & 1 & 0 \end{array}\right]$$

NOT INVERTIBLE.

b) $$\left[\begin{array}{cccc|cccc} 1 & 1 & 1 & 1 & 1 & 0 & 0 & 0 \\ 1 & 2 & 1 & 1 & 0 & 1 & 0 & 0 \\ 1 & 2 & 1 & 2 & 0 & 0 & 1 & 0 \\ 1 & 2 & 2 & 3 & 0 & 0 & 0 & 1 \end{array}\right]$$

$$\sim \left[\begin{array}{cccc|cccc} 1 & 1 & 1 & 1 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & -1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 1 & -1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 2 & -1 & 0 & 0 & 1 \end{array}\right]$$

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$$\sim \left[\begin{array}{cccc|cccc} 1 & 1 & 1 & 1 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & -1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & -1 & 1 & 0 \\ 0 & 0 & 1 & 2 & 0 & -1 & 0 & 1 \end{array}\right]$$

$$\sim \left[\begin{array}{cccc|cccc} 1 & 1 & 1 & 1 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & -1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 2 & 0 & -1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 & -1 & 1 & 0 \end{array}\right]$$

$$\left[\begin{array}{cccc|cccc} 1 & 1 & 1 & 0 & 1 & 1 & -1 & 0 \\ 0 & 1 & 0 & 0 & -1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 & -2 & 1 \\ 0 & 0 & 0 & 1 & 0 & -1 & 1 & 0 \end{array}\right]$$

$$\sim \left[\begin{array}{cccc|cccc} 1 & 1 & 0 & 0 & 1 & 0 & 1 & -1 \\ 0 & 1 & 0 & 0 & -1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 & -2 & 1 \\ 0 & 0 & 0 & 1 & 0 & -1 & 1 & 0 \end{array}\right]$$

$$\sim \left[\begin{array}{cccc|cccc} 1 & 0 & 0 & 0 & 2 & -1 & 1 & -1 \\ 0 & 1 & 0 & 0 & -1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 & -2 & 1 \\ 0 & 0 & 0 & 1 & 0 & -1 & 1 & 0 \end{array}\right] \text{(over)}$$

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so $\boxed{A^{-1} = \begin{pmatrix} 2 & -1 & 1 & -1 \\ -1 & 1 & 0 & 0 \\ 0 & 1 & -2 & 1 \\ 0 & -1 & 1 & 0 \end{pmatrix}}$

5) $AA^{-1} = I$

$\Rightarrow \det(AA^{-1}) = \det(I) = 1$

$\det(A)\det(A^{-1}) = 1$

$\therefore \det(A^{-1}) = \frac{1}{\det(A)}$

6) Suppose $C = AB$ invertible

$\Rightarrow \det(C) \neq 0$

$\det(AB) = \det(A) \cdot \det(B)$

$\Rightarrow \det(A) \neq 0$ & $\det(B) \neq 0$

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Suppose $\det(A) \neq 0$ & $\det(B) \neq 0$

$$\Rightarrow \det(C) = \det(AB) = \underbrace{\det(A)\det(B)}_{\neq 0}$$

But since $\det(C) \neq 0$

$\Rightarrow C$ is invertible.

I'm demonstrating the theorems

7a) $\det\begin{pmatrix} 2 & 1 \\ 3 & 4 \end{pmatrix} = 8-3 = \boxed{5}$

b) $\det\begin{pmatrix} 2 & 3 \\ 1 & 4 \end{pmatrix} = \det\begin{pmatrix} 2 & 1 \\ 3 & 4 \end{pmatrix}^T = 8-3 = \boxed{5}$

c) $\det\begin{pmatrix} 8 & 3 \\ 4 & 4 \end{pmatrix} = \det\begin{pmatrix} 4\cdot 2 & 3 \\ 4\cdot 1 & 4 \end{pmatrix}$

$= 4\det\begin{pmatrix} 2 & 3 \\ 1 & 4 \end{pmatrix}$

$= 4\cdot 5 = \boxed{20}$

d) $\det\begin{pmatrix} -2 & -1 \\ 3 & 4 \end{pmatrix} = \det\begin{pmatrix} -1\cdot 2 & -1\cdot 1 \\ 3 & 4 \end{pmatrix}$

$= -1\det\begin{pmatrix} 2 & 1 \\ 3 & 4 \end{pmatrix} = \boxed{-5}$

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3a) $\det\begin{pmatrix}1&2&1\\0&0&0\\1&1&2\end{pmatrix}=0$ B.C Row 2 = 0

b) $\det\begin{pmatrix}1&2&1\\0&1&0\\0&0&2\end{pmatrix}=$ $1\cdot1\cdot2=2$ B.C. matrix is upper Δ.

c) $\det\begin{pmatrix}2&0&0\\2&1&0\\1&1&2\end{pmatrix}=$ $2\cdot1\cdot2=4$ B.C matrix is lower Δ.

d) $\det\begin{pmatrix}3\cdot1&3\cdot2&3\cdot1\\3\cdot1&3\cdot1&3\cdot2\\3\cdot1&3\cdot1&3\cdot1\end{pmatrix}$

$=3^3\det\begin{pmatrix}1&2&1\\1&1&2\\1&1&1\end{pmatrix}$

R2=R2-R1
R3=R3-R1

$=3^3\det\begin{pmatrix}1&2&1\\0&-1&1\\0&-1&0\end{pmatrix}=$ (Ore!)

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$$= 3^3 \det\begin{pmatrix} 1 & 2 & 1 \\ 0 & -1 & 1 \\ 0 & 0 & -1 \end{pmatrix}$$ R3 = R3 − R2

$$= 27 \cdot ((1) \cdot (-1) \cdot (-1)) = \boxed{27}$$

e) $$\det\begin{pmatrix} 1 & 2 & 1 \\ 5 \cdot 1 & 5 \cdot 1 & 5 \cdot 2 \\ 1 & 1 & 1 \end{pmatrix}$$

$$= 5 \det\begin{pmatrix} 1 & 2 & 1 \\ 1 & 1 & 2 \\ 1 & 1 & 1 \end{pmatrix} = \boxed{5 \cdot 1}$$

f) $$\det\begin{pmatrix} 1 & 2 & 1 \\ 2+1 & 2+1 & 3+2 \\ 1 & 1 & 1 \end{pmatrix}$$

$$= \det\begin{pmatrix} 1 & 2 & 1 \\ 2 & 2 & 3 \\ 1 & 1 & 1 \end{pmatrix} + \det\begin{pmatrix} 1 & 2 & 1 \\ 1 & 1 & 2 \\ 1 & 1 & 1 \end{pmatrix}$$

$$= \det\begin{pmatrix} 1 & 2 & 1 \\ 0 & -2 & 1 \\ 0 & -1 & 0 \end{pmatrix} + 1$$ 1 from (d)

R2 = R2 − 2R1
R3 = R3 − R1

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$$= \det \begin{pmatrix} 1 & 2 & 1 \\ 0 & +1 & 0 \\ 0 & -2 & 1 \end{pmatrix} + 1$$

$R2 = -R2$
$R2 \leftrightarrow R3$

$$= \det \begin{pmatrix} 1 & 2 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} + 1$$

$$= \boxed{1 \cdot 1 \cdot 1 + 1 = 2}$$

You could have done any of these 3x3's by the crisscross formula. I did what I did here just to demonstrate the Theorems given on this assignment.