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# Linear Operators

1) Let $\mathcal{L}: \mathcal{V}_x \to \mathcal{V}_y$ be linear

a) Consider the set of vectors

$$\text{Null}(\mathcal{L}) = \{\vec{x} \in \mathcal{V}_x : \mathcal{L}(\vec{x}) = \vec{0}\}$$

Claim: Null($\mathcal{L}$) defines a subspace of $\mathcal{V}_x$.

Need only show Null($\mathcal{L}$) is closed wrt scalar mult and vect add in $\mathcal{V}_x$

Suppose $\alpha \in \mathcal{F}$ and $\vec{x} \in \text{Null}(\mathcal{L})$

Then $\mathcal{L}(\alpha\vec{x}) = \alpha\mathcal{L}(\vec{x}) = \alpha\vec{0} = \vec{0}$

$\Rightarrow \alpha\vec{x} \in \text{Null}(\mathcal{L})$

Suppose $\vec{x}_1$ and $\vec{x}_2 \in \text{Null}(\mathcal{L})$

Then $\mathcal{L}(\vec{x}_1 + \vec{x}_2) = \mathcal{L}(\vec{x}_1) + \mathcal{L}(\vec{x}_2) = \vec{0} + \vec{0} = \vec{0}$

$\Rightarrow \vec{x}_1 + \vec{x}_2 \in \text{Null}(\mathcal{L})$

So Null($\mathcal{L}$) is closed wrt scalar mult and vect add and can conclude Null($\mathcal{L}$) is a subspace of $\mathcal{V}_x$.

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b) consider a set of vectors

$$Rang(\mathcal{L}) = \{\vec{y} = \mathcal{L}(\vec{x}) : x \in V_x\}$$

Claim $Rang(\mathcal{L})$ defines a subspace of $V_y$.

Need only show $Rang(\mathcal{L})$ is closed wrt scaler mult and vect add on $V_y$.

Suppose $\alpha \in \mathcal{F}$ and $\vec{y} \in Rang(\mathcal{L})$

since $\vec{y} \in Rang(\mathcal{L})$ says $\exists\, x \in V_x$ such that $y = \mathcal{L}(x)$. But

$\alpha \vec{y} = \alpha \mathcal{L}(\vec{x}) = \mathcal{L}(\alpha \vec{x})$.

But $\alpha x \in V_x$ since $V_x$ is a vecspace

so $\alpha \vec{y} \in Rang(\mathcal{L})$

Suppose $\vec{y}_1$ and $\vec{y}_2 \in Rang(\mathcal{L})$

so $y_1 = \mathcal{L}(x_1)$ and $y_2 = \mathcal{L}(x_2)$

$\Rightarrow y_1 + y_2 = \mathcal{L}(x_1) + \mathcal{L}(x_2) = \mathcal{L}(x_1 + x_2)$

But $V_x$ is a vector space

which says $\vec{X}_1+\vec{X}_2 \in V_X$

So $Y_1+Y_2 \in Rang(\mathcal{L})$

$\therefore$ $Rang(\mathcal{L})$ is closed wrt scalar mult and vect add and we conclude $Rang(\mathcal{L})$ is a subspace of $V_Y$.

2 $\mathcal{L}: V \to V$ is linear and suppose $Null(\mathcal{L}) = \{\vec{0}\}$.

Claim $Rang(\mathcal{L}) = V$.

Rank Thrm says

$dim(Rang(\mathcal{L})) + dim(Null(\mathcal{L})) = dim(V)$

But since $Null(\mathcal{L}) = \{\vec{0}\}$ This says $dim(Null(\mathcal{L})) = 0$.

$\therefore$ $dim(Rang(\mathcal{L})) = dim(V)$

But also $Rang(\mathcal{L}) \subseteq V$ and since $dim(Rang(\mathcal{L})) = dim(V)$ (over)

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conclude $\text{Rang}(\mathcal{L}) = V$.

Now let $\vec{y} \in V = \text{Rang}(\mathcal{L})$

$\Rightarrow \exists\, x \in V$ such

that $\vec{y} = \mathcal{L}(\vec{x})$.

Claim $\vec{x}$ is unique. Suppose

There are two vectors $\vec{x}_1$ and $\vec{x}_2 \in V$

such that

$\vec{y} = \mathcal{L}(\vec{x}_1)$ and $y = \mathcal{L}(\vec{x}_2)$

$\Rightarrow \vec{0} = \vec{y} - \vec{y} = \mathcal{L}(\vec{x}_1) - \mathcal{L}(\vec{x}_2)$

$= \mathcal{L}(\vec{x}_1 - \vec{x}_2)$

$\Rightarrow \vec{x}_1 - \vec{x}_2 \in \text{Null}(\mathcal{L})$

$\Rightarrow \vec{x}_1 - \vec{x}_2 = \vec{0}$ because $\text{Null}(\mathcal{L}) = \{\vec{0}\}$

i.e. $\vec{x}_1 = \vec{x}_2$ and therefore the

soln is unique.

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3) Consider $A = \begin{pmatrix} 1 & 2 & 1 \\ 2 & 5 & 2 \\ 1 & 3 & 1 \end{pmatrix}$

a) To get Null(A) find all solns to $A\vec{x} = \vec{0}$

$$\left[\begin{array}{ccc|c} 1 & 2 & 1 & 0 \\ 2 & 5 & 2 & 0 \\ 1 & 3 & 1 & 0 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & 2 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{array}\right]$$

$$\sim \left[\begin{array}{ccc|c} 1 & 2 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

$x_3$ is free, say

$x_3 = \alpha$

$x_2 = 0$

$x_1 + 2x_2 + x_3 = 0$ says $x_1 = -2 0 - \alpha$

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} -\alpha \\ 0 \\ \alpha \end{pmatrix}$$

$$= \alpha \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}$$

So $\text{Null}(A) = \text{Span}\left\{ \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix} \right\}$ which BTW $= \text{Span}\left\{ \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix} \right\}$

b) $\text{Rang}(A) = \text{Span}\left\{ \begin{pmatrix}1\\2\\1\end{pmatrix}, \begin{pmatrix}2\\5\\3\end{pmatrix}, \begin{pmatrix}1\\2\\1\end{pmatrix} \right\}$

Let's compute "standard basis" for Rang(A)

$$\begin{bmatrix} 1 & 2 & 1 \\ 2 & 5 & 3 \\ 1 & 2 & 1 \end{bmatrix} \sim \begin{bmatrix} 1 & 2 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

Pivot

Free

But need to continue to get the augmented matrix to "full echelon" form

$$\sim \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

and read off standard basis

$\text{Rang}(A) = \text{Span}\left\{ \begin{pmatrix}1\\0\\-1\end{pmatrix}, \begin{pmatrix}0\\1\\1\end{pmatrix} \right\}$

4) Consider $A = \begin{pmatrix} 1 & 2 & 1 \\ 2 & 4 & 2 \\ 3 & 6 & 3 \end{pmatrix}$

a) To get null(A) find all solns to $A\vec{y} = \vec{0}$

$$\left[\begin{array}{ccc|c} 1 & 2 & 1 & 0 \\ 2 & 4 & 2 & 0 \\ 3 & 6 & 3 & 0 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & 2 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$ over

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So $x_2$ and $x_3$ are free variables

Say $x_2 = \alpha$, $x_3 = \beta$ so

$$x_1 + 2\alpha + \beta = 0 \implies x_1 = -2\alpha - \beta$$

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} -2\alpha - \beta \\ \alpha \\ \beta \end{pmatrix} = \alpha \begin{pmatrix} -2 \\ 1 \\ 0 \end{pmatrix} + \beta \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}$$

$$\therefore \text{Null}(A) = \text{Span}\left\{ \begin{pmatrix} -2 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix} \right\}$$

of course $= \text{Span}\left\{ \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}, \begin{pmatrix} -2 \\ 1 \\ 0 \end{pmatrix} \right\}$

b.) $\text{Rang}(A) = \underbrace{\text{Span}\left\{ \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}, \begin{pmatrix} 2 \\ 4 \\ 6 \end{pmatrix}, \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} \right\}}_{=(\text{column span})}$

No let's get its standard basis

Transpose

$$\begin{bmatrix} 1 & 2 & 3 \\ 2 & 4 & 6 \\ 1 & 2 & 3 \end{bmatrix} \sim \begin{bmatrix} 1 & 2 & 3 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

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read off the standard basis from this (its in "full echelon form")

$$Rang(A) = Span\left\{\begin{pmatrix}1\\2\\3\end{pmatrix}\right\}$$

Note $\dim(Rang(A)) + \dim(Null(A)) = \dim(\mathbb{R}^3)$

$$\underset{1}{\shortparallel} \quad + \quad \underset{2}{\shortparallel} \quad = 3 \checkmark$$

5a) $A = \begin{pmatrix}1 & 1 & 2\\ 2 & 2 & 4\\ 3 & 4 & 9\\ 4 & 2 & 2\end{pmatrix}$

a) $Null(A)$: Solve $A\vec{x} = \vec{0}$

$$\left[\begin{array}{ccc|c}1 & 1 & 2 & 0\\ 2 & 2 & 4 & 0\\ 3 & 4 & 9 & 0\\ 4 & 2 & 2 & 0\end{array}\right] \sim \left[\begin{array}{ccc|c}1 & 1 & 2 & 0\\ 0 & 0 & 0 & 0\\ 0 & 1 & 3 & 0\\ 0 & -2 & -6 & 0\end{array}\right]$$

$$\sim \left[\begin{array}{ccc|c}1 & 1 & 2 & 0\\ 0 & 1 & 3 & 0\\ 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0\end{array}\right]$$

$x_3$ is free, $x_3 = \alpha$

$x_2 + 3\alpha = 0 \quad x_2 = -3\alpha$

$x_1 + (-3\alpha) + 2\alpha = 0$

$x_1 = \alpha$

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$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} \alpha \\ -3\alpha \\ \alpha \end{pmatrix} = \alpha \begin{pmatrix} 1 \\ -3 \\ 1 \end{pmatrix}$$

$$\Rightarrow \text{Null}(A) = \text{Span}\left\{ \begin{pmatrix} 1 \\ -3 \\ 1 \end{pmatrix} \right\}$$

b) Rang(A); Transpose and reduce to full echelon form

$$\begin{bmatrix} 1 & 2 & 3 & 4 \\ 1 & 2 & 4 & 2 \\ 2 & 4 & 9 & 2 \end{bmatrix} \sim \begin{bmatrix} 1 & 2 & 3 & 4 \\ 0 & 0 & 1 & -2 \\ 0 & 0 & 3 & -6 \end{bmatrix}$$

$$\sim \begin{bmatrix} 1 & 2 & 3 & 4 \\ 0 & 0 & 1 & -2 \\ 0 & 0 & 0 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 2 & 0 & 10 \\ 0 & 0 & 1 & -2 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

full echelon form

and read off standard basis

$$\text{Rang}(A) = \text{Span}\left\{ \begin{pmatrix} 1 \\ 2 \\ 0 \\ 10 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \\ -2 \end{pmatrix} \right\}$$

c) $\dim(\text{Rang}(A)) + \dim(\text{Null}(A)) = \dim(\mathbb{R}^3)$

$$\underset{2}{\parallel} \quad + \quad 1 \quad = 3 \quad \checkmark$$

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6). $A = \begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 1 & 2 & 2 \end{pmatrix}$

a) Null(A); Solve $A\vec{x} = \vec{0}$

$$\left[\begin{array}{ccc|c} 1 & 1 & 1 & 0 \\ 0 & 1 & 1 & 0 \\ 1 & 2 & 2 & 0 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & 1 & 1 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 1 & 1 & 0 \end{array}\right]$$

$$\sim \left[\begin{array}{ccc|c} 1 & 1 & 1 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

$x_3 = \alpha$

$x_2 + \alpha = 0 \quad x_2 = -\alpha$

$x_1 + (-\alpha) + \alpha = 0 \quad x_1 = 0$

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ -\alpha \\ \alpha \end{pmatrix} = \alpha \begin{pmatrix} 0 \\ -1 \\ 1 \end{pmatrix}$$

So $\text{Null}(A) = \text{Span}\left\{\begin{pmatrix} 0 \\ -1 \\ 1 \end{pmatrix}\right\}$ which BTW $= \text{Span}\left\{\begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix}\right\}$

b) Rang(A): Transpose and reduce to full echelon form

$$\begin{bmatrix} 1 & 0 & 1 \\ 1 & 1 & 2 \\ 1 & 1 & 2 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 0 & 1 & 1 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

so read off standard basis

$$\text{Rang}(A) = \text{span}\left\{ \begin{pmatrix}1\\0\\1\end{pmatrix}, \begin{pmatrix}0\\1\\1\end{pmatrix} \right\}$$

c) $\dim(\text{Rang}(A)) + \dim(\text{Null}(A)) = \dim(\mathbb{R}^3)$

$\quad\quad\; 2 \quad\quad\quad + \quad\quad\quad 1 \quad\quad\quad = 3$ ✓

7) $A = \begin{pmatrix} 1 & 1 & 1 & 2 \\ 1 & 0 & 1 & 0 \\ 2 & 1 & 0 & 1 \end{pmatrix}$

a) Null(A) : Solve $A\vec{x} = \vec{0}$

$$\left[\begin{array}{cccc|c} 1 & 1 & 1 & 2 & 0 \\ 1 & 0 & 1 & 0 & 0 \\ 2 & 1 & 0 & 1 & 0 \end{array}\right] \sim \left[\begin{array}{cccc|c} 1 & 1 & 1 & 2 & 0 \\ 0 & 1 & 0 & 2 & 0 \\ 0 & -1 & -2 & -3 & 0 \end{array}\right]$$

$$\sim \left[\begin{array}{cccc|c} 1 & 1 & 1 & 2 & 0 \\ 0 & 1 & 0 & 2 & 0 \\ 0 & 0 & -2 & -1 & 0 \end{array}\right]$$

free variable

$x_4 = \alpha$

$-2x_3 - \alpha = 0 \quad x_3 = -\tfrac{1}{2}\alpha$

$x_2 + 2\alpha = 0 \quad x_2 = -2\alpha$

$x_1 - 2\alpha - \tfrac{1}{2}\alpha + 2\alpha = 0$

$x_1 = \tfrac{1}{2}\alpha$

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So $\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} \frac{1}{2}\alpha \\ -2\alpha \\ -\frac{1}{2}\alpha \\ \alpha \end{pmatrix} = \frac{\alpha}{2}\begin{pmatrix} 1 \\ -4 \\ -1 \\ 2 \end{pmatrix}$

$\Rightarrow \text{Null}(A) = \text{Span}\left\{\begin{pmatrix} 1 \\ -4 \\ -1 \\ 2 \end{pmatrix}\right\} \leftarrow$

b) Row(A) : Transpose and reduce to full echelon form

$$\begin{bmatrix} 1 & 1 & 2 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \\ 2 & 0 & 1 \end{bmatrix} \sim \begin{bmatrix} 1 & 1 & 2 \\ 0 & -1 & -1 \\ 0 & 0 & -2 \\ 0 & -2 & -3 \end{bmatrix}$$

$$\sim \begin{bmatrix} 1 & 1 & 2 \\ 0 & 1 & 1 \\ 0 & 2 & 3 \\ 0 & 0 & 1 \end{bmatrix} \sim \begin{bmatrix} 1 & 1 & 2 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \\ 0 & 0 & 1 \end{bmatrix} \sim \begin{bmatrix} 1 & 1 & 2 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

$$\sim \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

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0
0
0

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So read off standard basis

$$\text{Rang}(A) = \text{span}\left\{ \begin{pmatrix}1\\0\\0\end{pmatrix}, \begin{pmatrix}0\\1\\0\end{pmatrix}, \begin{pmatrix}0\\0\\1\end{pmatrix} \right\}$$

c): $\dim(\text{Rang}(A)) + \dim(\text{Null}(A)) = \dim(\mathbb{R}^4)$

$$\underset{3}{\shortparallel} + \underset{1}{\shortparallel} \quad \underset{4}{\shortparallel} \ \checkmark$$

8) $\mathcal{L}(p) \equiv \dfrac{d^2p}{dx^2} + \dfrac{dp}{dx}$ maps $P_2$ into $P_2$.

a) $p \in P_2 \quad p(x) = \alpha_0 + \alpha_1 x + \alpha_2 x^2$

$\mathcal{L}(p) = 0 = (2\alpha_2) + (\alpha_1) + 2\alpha_2 x$

$\Rightarrow \alpha_2 = 0 \Rightarrow \alpha_1 = 0$

So $p \in \text{Null}(\mathcal{L}) \Rightarrow p(x) = \alpha_0 \cdot 1$

So $\text{Null}(\mathcal{L}) = \text{span}\{1\}$.

The one function.

b) If you followed the proof of the rank theorem (see notes) see that

direct sum

$$\mathcal{P}_2 = \text{span}\{1\} \oplus \text{span}\{x, x^2\}$$
$$= \text{Null}(\mathcal{L}) \oplus M$$

So

$$\text{Range}(\mathcal{L}) = \text{span}\{\mathcal{L}(x), \mathcal{L}(x^2)\}$$
$$= \text{span}\{1, 2+2x\}$$
$$= \text{span}\{1, x+1\}$$

and this must be independent.

Here's another way to get the range of $\mathcal{L}$.

Let $p = \alpha_0 \cdot 1 + \alpha_1 x + \alpha_2 x^2$

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and unpule that

$$\mathcal{L}(p) = 2\alpha_2 + \alpha_1 + 2\alpha_2 X$$

$$= (2\alpha_2 + \alpha_1) + 2\alpha_2 X$$

Let $g$ be an arb poly of degree less than or equal to one, say $g(x) = \beta_0 \cdot 1 + \beta_1 X$. Then

$$\mathcal{L}(p) = g \Rightarrow (2\alpha_2 + \alpha_1) + 2\alpha_2 X = \beta_0 \cdot 1 + \beta_1 X$$

$$\Rightarrow 2\alpha_2 = \beta_1 \Rightarrow \alpha_2 = \beta_1 / 2$$

$$2\alpha_2 + \alpha_1 = \beta_0 \Rightarrow \alpha_1 = \beta_0 - \beta_1$$

$$\therefore \text{Rang}(\mathcal{L}) = \text{Span}\{1, x\}$$

Please note that

$$\text{Span}\{1, x+1\} = \text{Span}\{1, x\}$$