Matrices and Linear Systems

1a $A+B = \begin{pmatrix} 1 & 2 & 3 \\ 3 & 2 & 1 \end{pmatrix} + \begin{pmatrix} 1 & 1 & 1 \\ 2 & 2 & 2 \end{pmatrix}$

$= \begin{pmatrix} 1+1 & 2+1 & 3+1 \\ 3+2 & 2+2 & 1+2 \end{pmatrix} = \begin{pmatrix} 2 & 3 & 4 \\ 5 & 4 & 3 \end{pmatrix}$

b) $B+C = \begin{pmatrix} 1 & 1 & 1 \\ 2 & 2 & 2 \end{pmatrix} + \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}$ $\boxed{\text{NOT Defined}}$

c) $C+D = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} + \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 1+1 \\ 2+1 \\ 3+1 \end{pmatrix} = \begin{pmatrix} 2 \\ 3 \\ 4 \end{pmatrix}$

d) $A+D = \begin{pmatrix} 1 & 2 & 3 \\ 3 & 2 & 1 \end{pmatrix} + \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$ $\boxed{\text{NOT Defined}}$

2a) $AB = \begin{pmatrix} 1 & 2 & 3 \\ 3 & 2 & 1 \end{pmatrix} \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} = \begin{pmatrix} 1+4+9 \\ 3+4+3 \end{pmatrix}$

$= \begin{pmatrix} 14 \\ 10 \end{pmatrix}$

b) $AC = \begin{pmatrix} 1 & 2 & 3 \\ 3 & 2 & 1 \end{pmatrix} \begin{pmatrix} 1 & 2 & 3 \end{pmatrix}$ $\boxed{\text{NOT Defined}}$

②

c) $AD = \begin{pmatrix} 1 & 2 & 3 \\ 3 & 2 & 1 \end{pmatrix} \begin{pmatrix} 1 & 4 \\ 2 & 5 \\ 3 & 6 \end{pmatrix}$

$= \begin{pmatrix} 1+4+9 & 4+10+18 \\ 3+4+3 & 12+10+6 \end{pmatrix} = \begin{pmatrix} 14 & 32 \\ 10 & 28 \end{pmatrix}$

d) $CD = \begin{pmatrix} 1 & 2 & 3 \end{pmatrix} \begin{pmatrix} 1 & 4 \\ 2 & 5 \\ 3 & 6 \end{pmatrix}$

$= \begin{pmatrix} 1+4+9 & 4+10+18 \end{pmatrix} = \begin{pmatrix} 14 & 32 \end{pmatrix}$

e) $CB = \begin{pmatrix} 1 & 2 & 3 \end{pmatrix} \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} = \begin{pmatrix} 1+4+9 \end{pmatrix}$

$= \begin{pmatrix} 14 \end{pmatrix}$

f) $BC = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} \begin{pmatrix} 1 & 2 & 3 \end{pmatrix} = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 4 & 6 \\ 3 & 6 & 9 \end{pmatrix}$

g) $DA = \begin{pmatrix} 1 & 4 \\ 2 & 5 \\ 3 & 6 \end{pmatrix} \begin{pmatrix} 1 & 2 & 3 \\ 3 & 2 & 1 \end{pmatrix} =$ (over)

(3)

$$= \begin{pmatrix} 1+12 & 2+8 & 3+4 \\ 2+15 & 4+10 & 6+5 \\ 3+18 & 6+12 & 9+6 \end{pmatrix} = \begin{pmatrix} 13 & 10 & 7 \\ 17 & 14 & 11 \\ 21 & 18 & 15 \end{pmatrix}$$

h) $BD = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} \begin{pmatrix} 1 & 4 \\ 2 & 5 \\ 3 & 6 \end{pmatrix}$ $\boxed{\text{Not defined}}$

3) Prove $(A+B)C = AC + BC$ when either side is defined.

Suppose $A \in \mathbb{R}^{m_A \times n_A}$

$B \in \mathbb{R}^{m_B \times n_B}$

$C \in \mathbb{R}^{m_C \times n_C}$

Let's go left to right

$A+B$ defined $\Rightarrow$ $m_A = m_B = m$
$n_A = n_B = L$

$(A+B)C$ defined $\Rightarrow$ $m_C = L$

So $A \in \mathbb{R}^{m \times L}$, $B \in \mathbb{R}^{m \times L}$, $C \in \mathbb{R}^{L \times n_C}$

and the result, $(A+B)C \in \mathbb{R}^{m \times n_C}$ (over)

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$1 \le i \le m, \; 1 \le j \le n_c$

$((A+B)C)_{ij}$

$\sum_{k=1}^{L} (A+B)_{ik} C_{kj}$

$= \sum_{k=1}^{L} A_{ik} C_{kj} + \sum_{k=1}^{L} B_{ik} C_{kj}$

$= (AC)_{ij} + (BC)_{ij}$ ✓

Now, from right to left,

AC defined $\Rightarrow n_A = m_C \equiv L$

BC defined $\Rightarrow n_B = m_C \equiv L$

AC & BC defined $\Rightarrow AC \in \mathbb{R}^{m_A \times n_C}$

$\Rightarrow BC \in \mathbb{R}^{m_B \times n_C}$

So for
$AC + BC$ to be defined $\Rightarrow m_A = m_B = m$

so $A \in \mathbb{R}^{m \times L}$, $B \in \mathbb{R}^{m \times L}$, $C \in \mathbb{R}^{L \times n_C}$

(over)

(5)

so for $1 \le i \le m$, $1 \le j \le n_C$

$(AC)_{ij} + (BC)_{ij}$

$$= \sum_{k=1}^{L} A_{ik} C_{kj} + \sum_{k=1}^{L} B_{ik} C_{kj}$$

$$= \sum_{k=1}^{L} (A_{ik} + B_{ik}) C_{kj}$$

$$= \sum_{k=1}^{L} (A+B)_{ik} C_{kj}$$

$$= ((A+B)C)_{ij} \checkmark$$

4a) $(A+B)C$

$$= \left(\begin{pmatrix} 1 & 2 \\ 2 & 3 \\ 2 & 1 \end{pmatrix} + \begin{pmatrix} 2 & 0 \\ 3 & 4 \\ 1 & 1 \end{pmatrix} \right) \begin{pmatrix} 1 \\ 2 \end{pmatrix}$$

$$= \begin{pmatrix} 3 & 2 \\ 5 & 7 \\ 3 & 2 \end{pmatrix} \begin{pmatrix} 1 \\ 2 \end{pmatrix} = \begin{pmatrix} 7 \\ 19 \\ 7 \end{pmatrix} \text{(one)}$$

$AC + BC$

$$= \begin{pmatrix} 1 & 2 \\ 2 & 3 \\ 2 & 1 \end{pmatrix} \begin{pmatrix} 1 \\ 2 \end{pmatrix} + \begin{pmatrix} 2 & 0 \\ 3 & 4 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} 1 \\ 2 \end{pmatrix}$$

$$= \begin{pmatrix} 5 \\ 8 \\ 4 \end{pmatrix} + \begin{pmatrix} 2 \\ 11 \\ 3 \end{pmatrix} = \begin{pmatrix} 7 \\ 19 \\ 7 \end{pmatrix} \checkmark$$

b) $F(D+E)$

$$= \begin{pmatrix} 1 & 2 \end{pmatrix} \left(\begin{pmatrix} 1 & 2 & 1 \\ 2 & 1 & 2 \end{pmatrix} + \begin{pmatrix} 2 & 0 & 1 \\ 3 & 1 & 1 \end{pmatrix} \right)$$

$$= \begin{pmatrix} 1 & 2 \end{pmatrix} \begin{pmatrix} 3 & 2 & 2 \\ 5 & 2 & 3 \end{pmatrix} = \begin{pmatrix} 13 & 6 & 8 \end{pmatrix}$$

$FD + FE$ (over)

$$= \begin{pmatrix} 1 & 2 \end{pmatrix} \begin{pmatrix} 1 & 2 & 1 \\ 2 & 1 & 2 \end{pmatrix} + \begin{pmatrix} 1 & 2 \end{pmatrix} \begin{pmatrix} 2 & 0 & 1 \\ 3 & 1 & 1 \end{pmatrix}$$

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$$= \begin{pmatrix} 5 & 4 & 5 \end{pmatrix} + \begin{pmatrix} 8 & 2 & 3 \end{pmatrix}$$

$$= \begin{pmatrix} 13 & 6 & 8 \end{pmatrix} \checkmark$$

5) $x_1 - 2x_2 + x_3 = 0$

$2x_1 + x_2 - 3x_3 = 5$

$4x_1 - 7x_2 + x_3 = -1$

a) $$\begin{pmatrix} 1 & -2 & 1 \\ 2 & 1 & -3 \\ 4 & -7 & 1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 5 \\ -1 \end{pmatrix}$$

b) $$\left[\begin{array}{ccc|c} 1 & -2 & 1 & 0 \\ 2 & 1 & -3 & 5 \\ 4 & -7 & 1 & -1 \end{array}\right] \quad \begin{array}{l} R2 = R2 - 2R1 \\ R3 = R3 - 4R1 \end{array}$$

$$\sim \left[\begin{array}{ccc|c} 1 & -2 & 1 & 0 \\ 0 & 5 & -5 & 5 \\ 0 & 1 & -3 & -1 \end{array}\right] \quad R2 \to R2/5$$

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$$\sim \left[\begin{array}{ccc|c} 1 & -2 & 1 & 0 \\ 0 & 1 & -1 & 1 \\ 0 & 1 & -3 & -1 \end{array}\right] \quad R3 = R3 - R2$$

$$\sim \left[\begin{array}{ccc|c} 1 & -2 & 1 & 0 \\ 0 & 1 & -1 & 1 \\ 0 & 0 & -2 & -2 \end{array}\right] \quad R3 = R3/-2$$

This is upper Δ matrix

$$\left[\begin{array}{ccc|c} 1 & -2 & 1 & 0 \\ 0 & 1 & -1 & 1 \\ 0 & 0 & 1 & 1 \end{array}\right]$$

(i) $x_3 = 1$

$x_2 - x_3 = 1 \Rightarrow x_2 = 2$

$x_1 - 2x_2 + x_3 = 0 \quad x_1 = 4 - 1 = 3$

So
$$\boxed{\begin{array}{l} x_1 = 3 \\ x_2 = 2 \\ x_3 = 1 \end{array}}$$

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6)
$$2x_1 - x_2 + 3x_3 = 5$$
$$2x_1 + 2x_2 + 3x_3 = 7$$
$$-2x_1 + 3x_2 = -3$$

a)
$$\begin{pmatrix} 2 & -1 & 3 \\ 2 & 2 & 3 \\ -2 & 3 & 0 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 5 \\ 7 \\ -3 \end{pmatrix}$$

b)
$$\left[\begin{array}{ccc|c} 2 & -1 & 3 & 5 \\ 2 & 2 & 3 & 7 \\ -2 & 3 & 0 & -3 \end{array}\right] \quad \begin{array}{l} R2 = R2 - R1 \\ R3 = R3 + R1 \end{array}$$

$$\sim \left[\begin{array}{ccc|c} 2 & -1 & 3 & 5 \\ 0 & 3 & 0 & 2 \\ 0 & 2 & 3 & 2 \end{array}\right] \quad R3 = 3R3 - 2R2$$

$$\sim \left[\begin{array}{ccc|c} 2 & -1 & 3 & 5 \\ 0 & 3 & 0 & 2 \\ 0 & 0 & 9 & 2 \end{array}\right]$$

c) $x_3 = 2/9$ $\qquad 2x_1 - x_2 + 3x_3 = 5$

$3x_2 = 2 \quad x_2 = 2/3 \qquad 2x_1 - 2/3 + 2/3 = 5$

(over)

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So $\boxed{\begin{aligned} x_1 &= 5/2 \\ x_2 &= 2/3 \\ x_3 &= 2/9 \end{aligned}}$

7) $3x_1 - 4x_2 + 5x_3 = 7$
$-3x_1 + 4x_2 - 2x_3 = -1$
$6x_1 - 8x_2 + x_3 = -4$

a) $\begin{pmatrix} 3 & -4 & 5 \\ -3 & 4 & -2 \\ 6 & -8 & 1 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 7 \\ -1 \\ -4 \end{pmatrix}$

b) $\left[\begin{array}{ccc|c} 3 & -4 & 5 & 7 \\ -3 & 4 & -2 & -1 \\ 6 & -8 & 1 & -4 \end{array}\right]$ $R2 = R2 + R1$
$R3 = R3 - 2R1$

$\sim \left[\begin{array}{ccc|c} 3 & -4 & 5 & 7 \\ 0 & 0 & 3 & 6 \\ 0 & 0 & -9 & -18 \end{array}\right]$ $R2 = R2/2$
$R3 = R3 + 3R2$

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$$\sim \left[\begin{array}{ccc|c} 3 & -4 & 5 & 7 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

c) $x_3 = 2 \qquad 3x_1 - 4\alpha + 10 = 7$

$x_2 = \alpha \qquad 3x_1 = -3 + 4\alpha$

so $\boxed{\begin{aligned} x_1 &= 4/3\alpha - 1 \\ x_2 &= \alpha \\ x_3 &= 2 \end{aligned}}$

8)
$$x_1 + x_2 - 3x_3 = 4$$
$$2x_1 + x_2 - x_3 = 2$$
$$3x_1 + 2x_2 - 4x_3 = 7$$

a) $\begin{pmatrix} 1 & 1 & -3 \\ 2 & 1 & -1 \\ 3 & 2 & -4 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 4 \\ 2 \\ 7 \end{pmatrix}$

b) $\left[\begin{array}{ccc|c} 1 & 1 & -3 & 4 \\ 2 & 1 & -1 & 2 \\ 3 & 2 & -4 & 7 \end{array}\right]$ $R2 = R2 - 2R1$
$R3 = R3 - 3R1$
(over)

$$\sim \left[\begin{array}{ccc|c} 1 & 1 & -3 & 4 \\ 0 & -1 & 5 & -6 \\ 0 & -1 & 5 & -5 \end{array}\right] \quad \begin{array}{l} R2 = -R2 \\ R3 = R3 - R2 \end{array}$$

$$\sim \left[\begin{array}{ccc|c} 1 & 1 & -3 & 4 \\ 0 & 1 & -5 & 6 \\ 0 & 0 & 0 & 1 \end{array}\right]$$

c) But this says

$0x_1 + 0x_2 + 0x_3 = 1$

Impossible. — $\boxed{\text{NO solution}}$

9)
$$\begin{array}{c} \begin{array}{cccc} 2 & 4 & 2 & 4 \end{array} \\ \left[\begin{array}{cccc|c} 1 & 2 & 1 & 2 & 1 \\ 1 & 2 & 4 & 3 & 2 \\ 2 & 4 & -4 & 4 & 2 \end{array}\right] \end{array} \quad \begin{array}{l} R2 = R2 - R1 \\ R3 = R3 - 2R1 \end{array}$$

$$\sim \left[\begin{array}{cccc|c} 1 & 2 & 1 & 2 & 1 \\ 0 & 0 & 3 & 1 & 1 \\ 0 & 0 & -6 & 0 & 0 \end{array}\right] \quad \begin{array}{l} R2 \Leftrightarrow R3 \\ R3 = R3/-6 \end{array}$$

$$\sim \left[\begin{array}{cccc|c} 1 & 2 & 1 & 2 & 1 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 3 & 1 & 1 \end{array}\right] \quad R3 = R3 - 3R2$$

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$$\sim \left[\begin{array}{cccc|c} 1 & 2 & 1 & 2 & 1 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 \end{array}\right]$$

$x_4 = 1$ $\quad x_1 + 2\alpha + 0 + 2 = 1$

$x_3 = 0$ $\quad x_1 = -1 - 2\alpha$

$x_2 = \alpha$

so $\boxed{\begin{array}{l} x_1 = -1-2\alpha \\ x_2 = \alpha \\ x_3 = 0 \\ x_4 = 1 \end{array}}$

10) $$\left[\begin{array}{cccc|c} 1 & 1 & 2 & 1 & 3 \\ 1 & -1 & 1 & -1 & 2 \\ 3 & -1 & 4 & -5 & 7 \\ 2 & 0 & 3 & 0 & 5 \end{array}\right] \begin{array}{l} \\ R2 = R2 - R1 \\ R3 = R3 - 3R1 \\ R4 = R4 - 2R1 \end{array}$$

$$\sim \left[\begin{array}{cccc|c} 1 & 1 & 2 & 1 & 3 \\ 0 & -2 & -1 & -2 & -1 \\ 0 & -4 & -2 & -8 & -2 \\ 0 & -2 & -1 & -2 & -1 \end{array}\right] \begin{array}{l} \\ /-1 \\ /-2 \\ /-1 \end{array}$$

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$$\left[\begin{array}{cccc|c} 1 & 1 & 2 & 1 & 3 \\ 0 & 2 & 1 & 2 & 1 \\ 0 & 2 & 1 & 4 & 1 \\ 0 & 2 & 1 & 2 & 1 \end{array}\right] \begin{array}{l} \\ \\ R3 = R3 - R2 \\ R4 = R4 - R2 \end{array}$$

$$\sim \left[\begin{array}{cccc|c} 1 & 1 & 2 & 1 & 3 \\ 0 & 2 & 1 & 2 & 1 \\ 0 & 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

So $x_4 = 0$

$x_3 = \alpha$

$2x_2 + x_3 + 2x_4 = 1$

$2x_2 = 1 - \alpha \qquad x_2 = 1/2(1-\alpha)$

$x_1 + x_2 + 2x_3 + x_4 = 3$

$x_1 + \frac{1}{2}(1-\alpha) + 2\alpha = 3$

$x_1 = 5/2 - 3/2\,\alpha = \frac{1}{2}(5 - 3\alpha)$

$$\boxed{\begin{array}{ll} x_1 = 1/2(5-3\alpha) & x_3 = \alpha \\ x_2 = 1/2(1-\alpha) & x_4 = 0 \end{array}}$$