Introduction to Vector Spaces: Independence, Span and Basis

A *vector space*, or sometimes called a *linear space*, is an abstract system composed of a set of objects called vectors, an associated *field* of scalars, see e.g. [†], together with the operations of vector addition and scalar multiplication. Let V denote the set of vectors and $\mathcal{F}$ denote the field of scalars. Here I'll use bold lowercase Roman letters to signify vectors, e.g. $\mathbf{x} \in V$, and lowercase Greek letters to signify scalars, e.g. $\alpha \in \mathcal{F}$.

I'm going to list out now what properties vector addition and scalar multiplication are required to satisfy on a given vector space.

(a–0) For every $\mathbf{x}$ and $\mathbf{y} \in V$ we have $\mathbf{x} + \mathbf{y} \in V$.

(a–1) For every $\mathbf{x}$, $\mathbf{y}$ and $\mathbf{z} \in V$ we have $(\mathbf{x} + \mathbf{y}) + \mathbf{z} = \mathbf{x} + (\mathbf{y} + \mathbf{z})$.

(a–2) For every $\mathbf{x}$ and $\mathbf{y} \in V$ we have $\mathbf{x} + \mathbf{y} = \mathbf{y} + \mathbf{x}$.

(a–3) There is a vector $\mathbf{0} \in V$ such that $\mathbf{x} + \mathbf{0} = \mathbf{x}$ for every $\mathbf{x} \in V$.

(a–4) For every $\mathbf{x} \in V$ there is a vector $\widetilde{\mathbf{x}} \in V$ such that $\mathbf{x} + \widetilde{\mathbf{x}} = \mathbf{0}$.

(m–0) For every $\alpha \in \mathcal{F}$ and $\mathbf{x} \in V$ we have $\alpha\mathbf{x} \in V$.

(m–1) For every α and $\beta \in \mathcal{F}$ we have $\alpha(\beta\mathbf{x}) = (\alpha\beta)\mathbf{x}$ for every $\mathbf{x} \in V$.

(m–2) If $1 \in \mathcal{F}$ is the scalar field's multiplicative identity then $1\mathbf{x} = \mathbf{x}$ for any $\mathbf{x} \in V$.

(d–1) For every $\alpha \in \mathcal{F}$ and $\mathbf{x}$ and $\mathbf{y} \in V$ we have $\alpha(\mathbf{x} + \mathbf{y}) = \alpha\mathbf{x} + \alpha\mathbf{y}$.

(d–2) $(\alpha + \beta)\mathbf{x} = \alpha\mathbf{x} + \beta\mathbf{x}$

Let me say a few words about these items. (a–0) says the set V is *closed* under vector addition. (m–0) says V is also closed under scalar multiplication. (a–1) and (a–2) say vector addition must be *associative* and *commutative*. (a–3) says V must contain an *additive identity*. Property (a–4) says every vector in V has an *additive inverse* in V. (d–1) and (d–2) are required scalar–vector distributive properties.

When the scalar field $\mathcal{F} = \mathbb{R}$ (the field of real numbers) one calls the vector space a *real vector space*. When $\mathcal{F} = \mathbb{C}$ (the field of complex numbers) the vector space is called a *complex vector space*.

On this homework, and for much of the semester, we will focus on a particular vector space whose set of vectors V are comprised of column matrices with real entries

$$V \equiv \left\{ \mathbf{x} = \begin{pmatrix} x_1 \\ \vdots \\ x_m \end{pmatrix} : x_1 \in \mathbb{R}, \ldots, x_m \in \mathbb{R} \right\},$$

[†] https://en.wikipedia.org/wiki/Field_(mathematics)

and scalars given by the field of real numbers $\mathbb{R}$. Vector addition and scalar multiplication is defined exactly as done for matrices on your previous homework,

$$\mathbf{x} = \begin{pmatrix} x_1 \\ \vdots \\ x_m \end{pmatrix}, \ \mathbf{y} = \begin{pmatrix} y_1 \\ \vdots \\ y_m \end{pmatrix}, \ \alpha \in \mathbb{R} \quad \Rightarrow \quad \mathbf{x} + \mathbf{y} \equiv \begin{pmatrix} x_1 + y_1 \\ \vdots \\ x_m + y_m \end{pmatrix} \quad \text{and} \quad \alpha\mathbf{x} \equiv \begin{pmatrix} \alpha x_1 \\ \vdots \\ \alpha x_m \end{pmatrix}.$$

We call this system $\mathbb{R}^m$. Clearly on $\mathbb{R}^m$

$$\mathbf{0} = \begin{pmatrix} 0 \\ \vdots \\ 0 \end{pmatrix} \quad \text{and for } \mathbf{x} \in \mathbb{R}^m \text{ its additive inverse is } \ \widetilde{\mathbf{x}} = \begin{pmatrix} -x_1 \\ \vdots \\ -x_m \end{pmatrix}.$$

Check on your own that all properties (a–0) thru (d–2) listed above are satisfied by this system, and so $\mathbb{R}^m$ defines a vector space.

A vector space has several useful properties (beyond the ones listed above) which can be derived entirely from (a–*), (m–*) and (d–*). Next, I'll state and prove some of these.

(p–1) The additive identity on a vector space is unique. To see this is true, suppose $\mathbf{0}$ and $\mathbf{0}'$ are both additive identities from V. From (a–3) this says in particular $\mathbf{0}' + \mathbf{0} = \mathbf{0}'$ as well as $\mathbf{0} + \mathbf{0}' = \mathbf{0}$. These together with (a–2) gives $\mathbf{0}' = \mathbf{0}' + \mathbf{0} = \mathbf{0} + \mathbf{0}' = \mathbf{0}$ and so $\mathbf{0}' = \mathbf{0}$. Therefore, the additive identity is unique.

(p–2) The additive inverse of a vector from a vector space is unique. To see this is true, let $\mathbf{x} \in V$ be arbitrary and suppose $\widetilde{\mathbf{x}}$ and $\widetilde{\mathbf{x}}'$ are both additive inverses for $\mathbf{x}$. That is, property (a–4) says $\mathbf{x} + \widetilde{\mathbf{x}} = \mathbf{0}$ as well as $\mathbf{x} + \widetilde{\mathbf{x}}' = \mathbf{0}$. These, together with (a–3), (a–2) and (a–1) yield

$$\widetilde{\mathbf{x}}' = \widetilde{\mathbf{x}}' + \mathbf{0} = \widetilde{\mathbf{x}}' + (\mathbf{x} + \widetilde{\mathbf{x}}) = \widetilde{\mathbf{x}} + (\mathbf{x} + \widetilde{\mathbf{x}}') = \widetilde{\mathbf{x}} + \mathbf{0} = \widetilde{\mathbf{x}},$$

and so $\widetilde{\mathbf{x}}' = \widetilde{\mathbf{x}}$. Therefore, $\mathbf{x}$'s additive inverse is unique.

(p–3) If for a given $\mathbf{x} \in V$ there is a $\mathbf{z} \in V$ satisfying $\mathbf{x} + \mathbf{z} = \mathbf{x}$ then we must have $\mathbf{z} = \mathbf{0}$. To see this is true, (a–4) tells us there is a vector $\widetilde{\mathbf{x}} \in V$ such that $\mathbf{x} + \widetilde{\mathbf{x}} = \mathbf{0}$. Use this together with what's given, i.e. $\mathbf{x} = \mathbf{x} + \mathbf{z}$, as well as (a–1), (a–2) and (a–3), to find

$$\mathbf{0} = \mathbf{x} + \widetilde{\mathbf{x}} = (\mathbf{x} + \mathbf{z}) + \widetilde{\mathbf{x}} = \mathbf{z} + (\mathbf{x} + \widetilde{\mathbf{x}}) = \mathbf{z} + \mathbf{0} = \mathbf{z}.$$

Therefore, $\mathbf{z}$ must be the vector space's (unique) additive identity.

Three additional useful facts are given in the following.

(p–4) For a given vector space, let 0 denote its scalar field's additive identity, let 1 denote its multiplicative identity, and let -1 denote the additive inverse of the scalar 1. For any vector $\mathbf{x} \in V$ and any scalar $\alpha \in \mathcal{F}$ we have

$$\text{(i)} \ \ 0\,\mathbf{x} = \mathbf{0}, \quad \text{(ii)} \ \ \alpha\,\mathbf{0} = \mathbf{0}, \quad \text{(iii)} \ \ -1\,\mathbf{x} = \widetilde{\mathbf{x}},$$

where $\widetilde{\mathbf{x}}$ as usual denotes $\mathbf{x}$'s additive inverse. To see that (i) is true, properties (m–2) and (d–1) combine to say

$$\mathbf{x} + 0\,\mathbf{x} = 1\,\mathbf{x} + 0\,\mathbf{x} = (1+0)\,\mathbf{x} = 1\,\mathbf{x} = \mathbf{x}.$$

But (p–3) above tells us this implies $0\,\mathbf{x}\ (= \mathbf{z}) = \mathbf{0}$. (ii) follows from (i) and (m–1) as follows. $\alpha\,\mathbf{0} = \alpha\,(0\,\mathbf{0}) = (\alpha\,0)\,\mathbf{0} = 0\,\mathbf{0} = \mathbf{0}$. Finally, (iii) follows from (a–1), (a–2), (a–3), (a–4), (d–1) and (i) above by writing

$$\widetilde{\mathbf{x}} = \widetilde{\mathbf{x}} + \mathbf{0} = \widetilde{\mathbf{x}} + (1-1)\,\mathbf{x} = \widetilde{\mathbf{x}} + \mathbf{x} + (-1)\,\mathbf{x} = \mathbf{0} + (-1)\,\mathbf{x} = -1\,\mathbf{x}.$$

These additional properties can be very useful when doing vector algebra. For example:

$$\text{Given that}\quad \alpha_1\,\mathbf{x}_1 + \alpha_2\,\mathbf{x}_2 + \alpha_3\,\mathbf{x}_3 = \mathbf{0}, \quad \text{where } \alpha_1 \neq 0$$
$$\Rightarrow\quad \mathbf{x}_1 = \beta_2\,\mathbf{x}_2 + \beta_3\,\mathbf{x}_3, \quad \text{where } \beta_2 = -\alpha_2/\alpha_1,\ \ \beta_3 = -\alpha_3/\alpha_1.$$

This calculation is obviously valid on a simple vector space such as $\mathbb{R}^m$. While always true, it may not be so obvious in more abstract situations.

A *subspace* $\mathcal{S}$ of a parent vector space $\mathcal{V}$ is composed of a <u>nonempty</u> subset of the parent's vectors, say $S \subseteq V$. It also shares the same scalar field and notion of vector addition and scalar multiplication with its parent. But this is <u>not enough</u> to call $\mathcal{S}$ a subspace of $\mathcal{V}$. A subspace <u>must</u> also be a vector space on its own. However, it isn't necessary to check that all conditions (a–0) thru (d–2) are satisfied in order to accomplish this goal. One needs only to check that the subset S is closed under the inherited notion of vector addition and scalar multiplication. We'll package this well known fact into the following statement.

(ssp) Let $\mathcal{V}$ be a vector space where V denotes its set of vectors. Consider a nonempty subset $S \subseteq V$ and the system $\mathcal{S}$ defined by this subset of vectors together with $\mathcal{V}$'s scalars and its notion of vector addition and scalar multiplication. Then, $\mathcal{S}$ is a vector space provided the subset S is closed under <u>both</u> vector addition and scalar multiplication.

The notation $\mathcal{S} \subseteq \mathcal{V}$ is often used to signify that $\mathcal{S}$ is a subspace of $\mathcal{V}$. You are asked to prove statement (ssp) in an exercise below.

Here are three examples to help clarify what a subspace is. Consider a subset of vectors from the vector space $\mathbb{R}^2$

$$S = \{\mathbf{x} \in \mathbb{R}^2 : x_1 = 1\}.$$

Does this set define a subspace of $\mathbb{R}^2$? The answer is no. S is neither closed under vector addition nor scalar multiplication. For example

$$\mathbf{x} = \begin{pmatrix}1\\0\end{pmatrix} \in S,\ \mathbf{y} = \begin{pmatrix}1\\1\end{pmatrix} \in S, \quad \text{but}\quad \mathbf{x}+\mathbf{y} = \begin{pmatrix}2\\1\end{pmatrix} \notin S,$$

and also

$$\mathbf{x} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \in S, \;\; \alpha = 0 \in \mathbb{R} \;\; \text{but} \;\; \alpha\mathbf{x} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \notin S.$$

Here's a second example. Consider

$$S = \{\mathbf{x} \in \mathbb{R}^2 : x_1 > 0\}.$$

This set is closed under vector addition since for arbitrary $\mathbf{x} \in S$ and $\mathbf{y} \in S$

$$\mathbf{x} = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \in S, \;\; \mathbf{y} = \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} \in S \quad \Rightarrow \quad x_1 > 0, \; y_1 > 0 \quad \Rightarrow \quad \mathbf{x} + \mathbf{y} = \begin{pmatrix} x_1 + y_1 \\ x_2 + y_2 \end{pmatrix} \in S,$$

since $x_1 + y_1 > 0$. But it's not closed under scalar multiplication since for example

$$\mathbf{x} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \in S, \;\; \alpha = 0 \in \mathbb{R} \;\; \text{but} \;\; \alpha\mathbf{x} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \notin S.$$

Therefore, $S = \{\mathbf{x} \in \mathbb{R}^2 : x_1 > 0\}$ does not define a subspace of $\mathbb{R}^2$.

Here's a third example. Consider

$$S = \{\mathbf{x} \in \mathbb{R}^2 : x_1 + x_2 = 0\}.$$

This set is closed under vector addition since

$$\begin{aligned} \mathbf{x} = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \in S, \;\; \mathbf{y} = \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} \in S \quad &\Rightarrow \quad x_1 + x_2 = 0, \; y_1 + y_2 = 0 \\ &\Rightarrow \quad \mathbf{x} + \mathbf{y} = \begin{pmatrix} x_1 + y_1 \\ x_2 + y_2 \end{pmatrix} \in S. \end{aligned}$$

But $\mathbf{x} + \mathbf{y} \in S$ because $(x_1 + y_1) + (x_2 + y_2) = x_1 + x_2 + y_1 + y_2 = 0 + 0 = 0$. Moreover

$$\mathbf{x} = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \in S \quad \Rightarrow \quad x_1 + x_2 = 0 \quad \Rightarrow \quad \alpha \in \mathbb{R}, \;\; \alpha\mathbf{x} = \begin{pmatrix} \alpha x_1 \\ \alpha x_2 \end{pmatrix} \in S,$$

because $(\alpha x_1) + (\alpha x_2) = \alpha(x_1 + x_2) = \alpha\, 0 = 0$. So, S is also closed under scalar multiplication. Therefore, this third example does in fact define a subspace of $\mathbb{R}^2$.

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1. Prove the statement given in (ssp) is true. Do not assume the parent space $\mathcal{V}$ is $\mathbb{R}^m$ but is a general vector space. Hint: Properties (a–1), (a–2), (m–1), (m–2), (d–1) and (d–2) are obviously true because the vectors in $\mathcal{S}$ are a subset of the vectors in $\mathcal{V}$. Properties (a–0) and (m–0) are assumed in the statement itself. You need to verify properties (a–3) and (a–4) are true. That is, show that $\mathbf{0} \in S$ and for any $\mathbf{x} \in S$ its additive inverse $\widetilde{\mathbf{x}} \in S$.

2. Determine whether or not the following sets define a subspace of $\mathbb{R}^2$.

(a) $\{\mathbf{x} \in \mathbb{R}^2 : x_1 = 0\}$ (c) $\{\mathbf{x} \in \mathbb{R}^2 : x_1 + x_2 \geq 0\}$

(b) $\{\mathbf{x} \in \mathbb{R}^2 : x_1 - x_2 = 0\}$ (d) $\{\mathbf{x} \in \mathbb{R}^2 : x_1 + 2x_2 = 0\}$

Either prove the set is closed under both vector addition and scalar multiplication or give an example to show it is not.

Consider a set of n vectors $\{\mathbf{x}_1, \ldots, \mathbf{x}_n\}$. This set is called a *dependent set* if there are n scalars, $\alpha_1, \ldots, \alpha_n$, which are <u>not</u> all zero such that

$$\alpha_1 \mathbf{x}_1 + \cdots + \alpha_n \mathbf{x}_n = \mathbf{0}.$$

A set of vectors that is not dependent is called an *independent set*.

Given that the vectors in the set above come from the vector space $\mathbb{R}^m$, we can use matrix elimination to determine whether the set is independent or not. The problem can be recast as follows.

$$\alpha_1 \mathbf{x}_1 + \cdots + \alpha_n \mathbf{x}_n = \mathbf{0} \quad \Longleftrightarrow \quad \begin{pmatrix} x_{1,1} & \cdots & x_{1,n} \\ \vdots & & \vdots \\ x_{m,1} & \cdots & x_{m,n} \end{pmatrix} \begin{pmatrix} \alpha_1 \\ \vdots \\ \alpha_n \end{pmatrix} = \begin{pmatrix} 0 \\ \vdots \\ 0 \end{pmatrix}.$$

Make sure to understand why the matrix multiplication formulation on the right is the same as what's written on the left. Check that the jth column of the $m \times n$ matrix on the right is the column vector $\mathbf{x}_j \in \mathbb{R}^m$. The zero matrix on the right has size $m \times 1$. Now see that if the <u>only</u> solution to this linear system is $\alpha_1 = \cdots = \alpha_n = 0$, then the set is <u>independent</u>. If the system has a *nontrivial solution*, however, the set is <u>dependent</u>.

Consider the following four vectors from $\mathbb{R}^3$.

$$\mathbf{x}_1 \equiv \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}, \ \mathbf{x}_2 \equiv \begin{pmatrix} 4 \\ 5 \\ 6 \end{pmatrix}, \ \mathbf{x}_3 \equiv \begin{pmatrix} 7 \\ 8 \\ 9 \end{pmatrix}, \ \mathbf{x}_4 \equiv \begin{pmatrix} 7 \\ 8 \\ 3 \end{pmatrix}.$$

I'm going to use these in the next two examples.

Is the set $\{\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3\}$ an independent set? The augmented matrix to consider is

$$\left[\begin{array}{ccc|c} 1 & 4 & 7 & 0 \\ 2 & 5 & 8 & 0 \\ 3 & 6 & 9 & 0 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & 4 & 7 & 0 \\ 0 & -3 & -6 & 0 \\ 0 & -6 & -12 & 0 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & 4 & 7 & 0 \\ 0 & 1 & 2 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right].$$

Back substitution tells us $\alpha_3 = \alpha$, $\alpha_2 = -2\alpha$ and $\alpha_1 = -4(-2\alpha) - 7(\alpha) = \alpha$ for any real number α. WLOG take $\alpha = 1$ to see $1\mathbf{x}_1 - 2\mathbf{x}_2 + 1\mathbf{x}_3 = \mathbf{0}$, and conclude $\{\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3\}$ is <u>not</u> an independent set of vectors.

Is the set $\{\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_4\}$ an independent set? The augmented matrix to consider here is

$$\left[\begin{array}{ccc|c} 1 & 4 & 7 & 0 \\ 2 & 5 & 8 & 0 \\ 3 & 6 & 3 & 0 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & 4 & 7 & 0 \\ 0 & -3 & -6 & 0 \\ 0 & -6 & -18 & 0 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & 4 & 7 & 0 \\ 0 & 1 & 2 & 0 \\ 0 & 1 & 3 & 0 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & 4 & 7 & 0 \\ 0 & 1 & 2 & 0 \\ 0 & 0 & 1 & 0 \end{array}\right].$$

This time back substitution tells us $\alpha_3 = 0$, $\alpha_2 = 0$ and $\alpha_1 = 0$. Therefore

$$\alpha_1 \mathbf{x}_1 + \alpha_2 \mathbf{x}_2 + \alpha_3 \mathbf{x}_4 = \mathbf{0} \quad \Rightarrow \quad \alpha_1 = \alpha_2 = \alpha_3 = 0,$$

and we conclude $\{\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_4\}$ <u>is</u> an independent set of vectors.

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3. Prove the following. A set of vectors $\{\mathbf{x}_1, \ldots, \mathbf{x}_n\}$ (assume $n \geq 2$) is dependent if and only if at least one its vectors can be written as a linear combination of the others. You need not assume these vectors come from $\mathbb{R}^m$. Hint: Show $\mathbf{x}_{i_*} = \sum_{k \neq i_*} \alpha_k \mathbf{x}_k$ for some index $1 \leq i_* \leq n$.

Consider the following vectors from $\mathbb{R}^4$.

$$\mathbf{x}_1 = \begin{pmatrix} 1 \\ 4 \\ 2 \\ -3 \end{pmatrix}, \quad \mathbf{x}_2 = \begin{pmatrix} 7 \\ 10 \\ -4 \\ -1 \end{pmatrix}, \quad \mathbf{x}_3 = \begin{pmatrix} -2 \\ 1 \\ 5 \\ -14 \end{pmatrix}, \quad \mathbf{x}_4 = \begin{pmatrix} -2 \\ 1 \\ 5 \\ -4 \end{pmatrix}.$$

4. Is $\{\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3\}$ an independent set of vectors?

5. Is $\{\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_4\}$ an independent set of vectors?

6. Is $\{\mathbf{x}_1, \mathbf{x}_3, \mathbf{x}_4\}$ an independent set of vectors?

Consider a finite set of vectors, $\{\mathbf{x}_1, \ldots, \mathbf{x}_n\}$, from a vector space $\mathcal{V}$. The *span* of this set is the subspace of $\mathcal{V}$ defined by

$$\text{span}\{\mathbf{x}_1, \ldots, \mathbf{x}_n\} \equiv \{\textstyle\sum_{k=1}^n \alpha_k \mathbf{x}_k : \text{each } \alpha_k \in \mathcal{F}\}$$

where $\mathcal{F}$ is $\mathcal{V}$'s scalar field. That is,

$$\mathbf{y} \in \text{span}\{\mathbf{x}_1, \ldots, \mathbf{x}_n\} \quad \Longleftrightarrow \quad \mathbf{y} = \alpha_1 \mathbf{x}_1 + \cdots + \alpha_n \mathbf{x}_n$$

for some collection of scalars $\alpha_1, \ldots, \alpha_n$. In other words, a vector is in $\text{span}\{\mathbf{x}_1, \ldots, \mathbf{x}_n\}$ when it can be written as a *linear combination* of the specified vectors $\mathbf{x}_1, \ldots, \mathbf{x}_n$. Make sure to convince yourself that $\text{span}\{\mathbf{x}_1, \ldots, \mathbf{x}_n\}$ really is closed under both vector addition and scalar multiplication, and therefore conclude the span is a subspace of $\mathcal{V}$ regardless of the particulars of the set of vectors which defines it.

If $\{\mathbf{x}_1, \ldots, \mathbf{x}_n\}$ is an <u>independent</u> set of vectors and $\mathbf{y} \in \text{span}\{\mathbf{x}_1, \ldots, \mathbf{x}_n\}$ then the decomposition $\mathbf{y} = \alpha_1 \mathbf{x}_1 + \cdots + \alpha_n \mathbf{x}_n$ is <u>unique</u>. Let me show you why. Suppose there are two ways to decompose $\mathbf{y}$, say

$$\begin{aligned} & \mathbf{y} = \alpha_1 \mathbf{x}_1 + \cdots + \alpha_n \mathbf{x}_n \quad \text{and} \quad \mathbf{y} = \beta_1 \mathbf{x}_1 + \cdots + \beta_n \mathbf{x}_n \\ \Rightarrow \quad & \mathbf{0} = (\alpha_1 - \beta_1)\, \mathbf{x}_1 + \cdots + (\alpha_n - \beta_n)\, \mathbf{x}_n \\ \Rightarrow \quad & (\alpha_1 - \beta_1) = \cdots = (\alpha_n - \beta_n) = 0. \end{aligned}$$

This last step follows from the fact that $\{\mathbf{x}_1, \ldots, \mathbf{x}_n\}$ is an independent set. So, since we have $\alpha_k = \beta_k$ for each $k = 1, \ldots, n$, the two decompositions above are in fact identical.

It's not hard to show the following. If $\{\mathbf{x}_1, \ldots, \mathbf{x}_n\}$ is a dependent set of vectors and $\mathbf{y} \in \text{span}\{\mathbf{x}_1, \ldots, \mathbf{x}_n\}$ then the decomposition $\mathbf{y} = \alpha_1 \mathbf{x}_1 + \cdots + \alpha_n \mathbf{x}_n$ is not unique. You are asked to show this in exercise 7 below.

Now lets again restrict our attention to the special vector space $\mathcal{V} = \mathbb{R}^m$. How do we compute whether or not a given vector is in a span? We'll use elimination of course. Consider the subspace $\mathcal{S} \equiv \text{span}\{\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3\} \subseteq \mathbb{R}^4$ where

$$\mathbf{x}_1 = \begin{pmatrix} 1 \\ 4 \\ 2 \\ -3 \end{pmatrix}, \quad \mathbf{x}_2 = \begin{pmatrix} 7 \\ 10 \\ -4 \\ -1 \end{pmatrix}, \quad \mathbf{x}_3 = \begin{pmatrix} -2 \\ 1 \\ 5 \\ -4 \end{pmatrix}.$$

Is $\mathbf{y} \equiv \begin{pmatrix} 9 \\ 27 \\ 9 \\ -17 \end{pmatrix} \in \text{span}\{\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3\}$? The linear system we have to solve is

$$\begin{pmatrix} 1 & 7 & -2 \\ 4 & 10 & 1 \\ 2 & -4 & 5 \\ -3 & -1 & -4 \end{pmatrix} \begin{pmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_3 \end{pmatrix} = \begin{pmatrix} 9 \\ 27 \\ 9 \\ -17 \end{pmatrix} \quad \Rightarrow \quad \left[\begin{array}{ccc|c} 1 & 7 & -2 & 9 \\ 4 & 10 & 1 & 27 \\ 2 & -4 & 5 & 9 \\ -3 & -1 & -4 & -17 \end{array}\right],$$

and we eliminate the augmented matrix to obtain

$$\sim \left[\begin{array}{ccc|c} 1 & 7 & -2 & 9 \\ 0 & 2 & -1 & 1 \\ 0 & 2 & -1 & 1 \\ 0 & 2 & -1 & 1 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & 7 & -2 & 9 \\ 0 & 2 & -1 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right].$$

Now, use back substitution. See that α_3 is a free variable, so let $\alpha_3 = \alpha$ where α is any real number. Then, $\alpha_2 = \frac{1}{2}(1 + \alpha)$ and $\alpha_1 = \frac{1}{2}(11 - 3\alpha)$. So we get

$$\mathbf{y} = \begin{pmatrix} 9 \\ 27 \\ 9 \\ -17 \end{pmatrix} = \tfrac{1}{2}(11 - 3\alpha)\,\mathbf{x}_1 + \tfrac{1}{2}(1 + \alpha)\,\mathbf{x}_2 + \alpha\,\mathbf{x}_3 \in \mathcal{S}.$$

Therefore we see $\mathbf{y} \in \mathcal{S}$. Moreover, since the decomposition is not unique, i.e. α here can be any real number, we also conclude the set of vectors $\{\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3\}$ is not independent. (Look back at exercise 5 above.)

Let me change $\mathbf{y}$ by a little bit and ask the same question.

Is $\mathbf{y} \equiv \begin{pmatrix} 9 \\ 27 \\ 9 \\ -16 \end{pmatrix} \in \text{span}\{\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3\}$? The augmented matrix to consider here is

$$\left[\begin{array}{rrr|r} 1 & 7 & -2 & 9 \\ 4 & 10 & 1 & 27 \\ 2 & -4 & 5 & 9 \\ -3 & -1 & -4 & -16 \end{array}\right] \sim \left[\begin{array}{rrr|r} 1 & 7 & -2 & 9 \\ 0 & 2 & -1 & 1 \\ 0 & 2 & -1 & 1 \\ 0 & 2 & -1 & 11/10 \end{array}\right] \sim \left[\begin{array}{rrr|r} 1 & 7 & -2 & 9 \\ 0 & 2 & -1 & 1 \\ 0 & 0 & 0 & 1/10 \\ 0 & 0 & 0 & 0 \end{array}\right].$$

However, the third row in the right above says $0\,\alpha_1 + 0\,\alpha_2 + 0\,\alpha_3 = 1/10$, and this is impossible. Therefore, this time $\mathbf{y} \notin \text{span}\{\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3\}$.

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7. Suppose $\{\mathbf{x}_1, \ldots, \mathbf{x}_n\}$ is a <u>dependent</u> set of vectors from a general vector space $\mathcal{V}$, and suppose $\mathbf{y} \in \text{span}\{\mathbf{x}_1, \ldots, \mathbf{x}_n\}$. Prove there are an <u>infinite</u> number decompositions such that $\mathbf{y} = \alpha_1\mathbf{x}_1 + \cdots + \alpha_n\mathbf{x}_n$.
Hint. Since $\{\mathbf{x}_1, \ldots, \mathbf{x}_n\}$ is a dependent set, there are scalars $\beta_1, \ldots, \beta_n$ which are not all zero such that $\beta_1\mathbf{x}_1 + \cdots + \beta_n\mathbf{x}_n = \mathbf{0}$.

8. Let $\{\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3\}$ come from exercise 4 above. Determine if the given vector $\mathbf{y}$ is in $\text{span}\{\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3\}$. If it is, write down and check the decomposition $\mathbf{y} = \alpha_1\mathbf{x}_1 + \alpha_2\mathbf{x}_2 + \alpha_3\mathbf{x}_3$.

(a) $\mathbf{y} = \begin{pmatrix} 1 \\ 2 \\ 3 \\ 4 \end{pmatrix}$ (b) $\mathbf{y} = \begin{pmatrix} 6 \\ 15 \\ 3 \\ -18 \end{pmatrix}$

A *basis* for a vector space is a linearly independent spanning set.

That is, $\{\mathbf{b}_1, \ldots, \mathbf{b}_n\}$ is a basis for a vector space $\mathcal{V}$ if:

(1) $\mathcal{V} = \text{span}\{\mathbf{b}_1, \ldots, \mathbf{b}_n\}$.

(2) $\{\mathbf{b}_1, \ldots, \mathbf{b}_n\}$ is an independent set.

The *dimension* of a vector space is the number of basis vectors needed to span it. It's not obvious, but this number is independent of any particular spanning basis.

Clearly,

$$\mathbb{R}^2 = \text{span}\{\mathbf{e}_1, \mathbf{e}_2\}, \quad \text{where } \mathbf{e}_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad \mathbf{e}_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix},$$

and so $\mathbb{R}^2$ is two dimensional (duh). Not as obvious, but you can check that this is another basis for $\mathbb{R}^2$

$$\mathbb{R}^2 = \text{span}\{\mathbf{b}_1, \mathbf{b}_2\}, \quad \text{where } \mathbf{b}_1 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \quad \mathbf{b}_2 = \begin{pmatrix} 2 \\ 3 \end{pmatrix}.$$

The basis $\{\mathbf{e}_1, \mathbf{e}_2\}$ is called the *standard basis* for $\mathbb{R}^2$. The standard basis for $\mathbb{R}^m$ is

$$\mathbf{e}_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix}, \quad \mathbf{e}_2 = \begin{pmatrix} 0 \\ 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix}, \quad \cdots, \quad \mathbf{e}_{m-1} = \begin{pmatrix} 0 \\ \vdots \\ 0 \\ 1 \\ 0 \end{pmatrix}, \quad \mathbf{e}_m = \begin{pmatrix} 0 \\ \vdots \\ 0 \\ 0 \\ 1 \end{pmatrix}.$$

One might think that the standard basis for $\mathbb{R}^m$ is the most useful of all of its bases. But it really depends on the application. Later in this course we will consider others.

Let me close out this assignment by showing you, by example, how to convert a given basis for a subspace of $\mathbb{R}^m$ to its standard basis.

Recall from exercise 4 you showed

$$\mathbf{x}_1 = \begin{pmatrix} 1 \\ 4 \\ 2 \\ -3 \end{pmatrix}, \quad \mathbf{x}_2 = \begin{pmatrix} 7 \\ 10 \\ -4 \\ -1 \end{pmatrix}, \quad \mathbf{x}_3 = \begin{pmatrix} -2 \\ 1 \\ 5 \\ -14 \end{pmatrix},$$

is an independent set. Therefore $\{\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3\}$ is a basis for $\mathcal{S} \equiv \text{span}\{\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3\}$. To determine $\mathcal{S}$'s standard basis, write out an augmented matrix using these three column vectors as rows

$$\begin{bmatrix} 1 & 4 & 2 & -3 \\ 7 & 10 & -4 & -1 \\ -2 & 1 & 5 & -14 \end{bmatrix}.$$

Notice there's no vertical bar ($|$) here. Now, row reduce to row echelon form

$$\sim \begin{bmatrix} 1 & 4 & 2 & -3 \\ 0 & 18 & 18 & -20 \\ 0 & 9 & 9 & -20 \end{bmatrix} \sim \begin{bmatrix} 1 & 4 & 2 & -3 \\ 0 & 18 & 18 & -20 \\ 0 & 0 & 0 & 20 \end{bmatrix} \sim \begin{bmatrix} 1 & 4 & 2 & -3 \\ 0 & 1 & 1 & -20/18 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Notice on the right I've scaled all pivots to one. Finally, starting from the right most pivot, use *backward elimination* to get

$$\sim \begin{bmatrix} 1 & 4 & 2 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & -2 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}.$$

This is called the *row canonical form* or alternatively the *reduced row echelon form* for the augmented matrix; see [‡] where the reduced row echelon form is discussed. The standard basis for the subspace $\mathcal{S} \equiv \text{span}\{\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3\}$ can now be read off as follows

$$\mathcal{S} = \text{span}\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}, \quad \text{where} \quad \mathbf{e}_1 = \begin{pmatrix} 1 \\ 0 \\ -2 \\ 0 \end{pmatrix}, \quad \mathbf{e}_2 = \begin{pmatrix} 0 \\ 1 \\ 1 \\ 0 \end{pmatrix}, \quad \mathbf{e}_3 = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}.$$

[‡] https://wikipedia.org/wiki/Row_echelon_form

BTW. I checked my calculation by observing

$$\begin{aligned}
\mathbf{x}_1 &= \quad \mathbf{e}_1 + 4\,\mathbf{e}_2 - 3\,\mathbf{e}_3 \\
\mathbf{x}_2 &= \quad 7\,\mathbf{e}_1 + 10\,\mathbf{e}_2 - \mathbf{e}_3 \\
\mathbf{x}_3 &= -2\,\mathbf{e}_1 + \mathbf{e}_2 - 14\,\mathbf{e}_3.
\end{aligned}$$

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9. Find the standard basis for $\operatorname{span}\{\mathbf{x}_1, \mathbf{x}_3, \mathbf{x}_4\}$ from exercise 6.

10. The set $\{\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_4\}$ from exercise 5 is not independent. However, it's still possible to determine the standard basis for $\operatorname{span}\{\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_4\}$ as just done. You'll get a zero row when eliminating to row canonical form. Disregard the zero row when you read off your basis. What is the dimension of $\operatorname{span}\{\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_4\}$? Answer: two.
