0

Linear Alg Part II

1a) Let X_* be any element of S

Then $X_* = 1X_*$ by m-2

$$= (1+0)X_*$$

$$= 1X_* + 0X_* \text{ by d-2}$$

$$= X_* + 0X_* \text{ by m-2}$$

So $X_* = X_* + 0X_*$

Also $0X_* \in S$ by m-0

Clearly for any $X \in S$ we have $X + 0X_* = X$ so $0X_* = \vec{0}$.

b) Let X be any element of S.

$\vec{0} = 0X$ by part 1a.

$$= (1-1)X$$

$$= 1X + (-1)X \text{ by d-2}$$

$$= X + (-1)X \text{ by m-2}$$

So $X' = (-1)X \in S$ by m-0.

②

2a) Let $S = \{ x \in \mathbb{R}^2 : x_1 = 0 \}$

Suppose $\vec{x} = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \in S$ & $\vec{y} = \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} \in S$

$\Rightarrow x_1 = 0$ and $y_1 = 0$

but $\vec{x} + \vec{y} = \begin{pmatrix} x_1 + y_1 \\ x_2 + y_2 \end{pmatrix}$ and

$x_1 + y_1 = 0 + 0 = 0 \Rightarrow \vec{x} + \vec{y} \in S$

So S is closed under addition.

Now, let $\alpha \in \mathbb{R}$

$\alpha \vec{x} = \begin{pmatrix} \alpha x_1 \\ \alpha x_2 \end{pmatrix}$

but $\alpha x_1 = \alpha \cdot 0 = 0 \Rightarrow \alpha \vec{x} \in S$

So since S is closed under both vector addition and scalar mult conclude S is a subspace of $\mathbb{R}^2$

③

b) $S = \{ X \in \mathbb{R}^2 : x_1 - x_2 = 0 \}$

let $\vec{x} = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \in S$ and $\vec{y} \in \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} \in S$

$\Rightarrow x_1 - x_2 = 0$ and $y_1 - y_2 = 0$

But $\vec{x} + \vec{y} = \begin{pmatrix} x_1 + y_1 \\ x_2 + y_2 \end{pmatrix}$ is in S

because

$$(x_1 + y_1) - (x_2 + y_2) = (x_1 - x_2) + (y_1 - y_2)$$
$$= 0 + 0 = 0$$

Also, let $\alpha \in \mathbb{R}$ then

$$\alpha \vec{x} = \begin{pmatrix} \alpha x_1 \\ \alpha x_2 \end{pmatrix}$$

But $\alpha x_1 - \alpha x_2 = \alpha (x_1 - x_2) = 0$

and so $\alpha \vec{x} \in S$

So S is closed under both add and mult $\Rightarrow$ S is a subspace of $\mathbb{R}^2$.

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c) $S = \{ X \in \mathbb{R}^2 : x_1 + x_2 \geq 0 \}$

Clearly $\vec{x} = \begin{pmatrix} 1 \\ 1 \end{pmatrix} \in S$

and $\alpha = -1 \in \mathbb{R}$

But $-1 \begin{pmatrix} 1 \\ 1 \end{pmatrix} = \begin{pmatrix} -1 \\ -1 \end{pmatrix} \notin S$

because $-1 + -1 = -2 \not\geq 0$.

So This S is NOT a subspace of $\mathbb{R}^2$.

d) $S = \{ \vec{X} \in \mathbb{R}^2 : x_1 + 2x_2 = 0 \}$

$X = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \in S$, $y = \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} \in S$

$\Rightarrow x_1 + 2x_2 = 0$ and $y_1 + 2y_2 = 0$

$X + y = \begin{pmatrix} x_1 + y_1 \\ x_2 + y_2 \end{pmatrix}$ is in S b.c.

$(x_1 + y_1) + 2(x_2 + y_2) = x_1 + 2x_2 + y_1 + 2y_2$
$= 0 + 0 = 0$.

(over)

(5)

Also, for any $\alpha \in \mathbb{R}$

$\alpha X = \begin{pmatrix} \alpha x_1 \\ \alpha x_2 \end{pmatrix} \in S$ b.c.

$\alpha x_1 + 2(\alpha x_2) = \alpha(x_1 + 2x_2) = \alpha \cdot 0 = 0$

so S is closed under both vec addition and scaler mult so S is a subspace of $\mathbb{R}^2$.

3) If $\{x_1, \ldots x_n\}$ is dependent

There are numbers $\alpha_1 \ldots \alpha_n$ which are <u>not</u> all zero such that $\alpha_1 x_1 + \ldots + \alpha_n x_n = 0$

Suppose $\alpha_{i\alpha} \neq 0$ (since at least one is nonzero)

Then
$$\alpha_{i\alpha} \vec{X}_{i\alpha} = -\sum_{k \neq i\alpha} \alpha_k \vec{X}_k$$

$$\Rightarrow X_{i\alpha} = \frac{1}{\alpha_{i\alpha}}\left(-\sum_{k \neq i\alpha} \alpha_k \vec{X}_k\right)$$ (over)

⑥

$$= \sum_{k \neq i_*} \tilde{\alpha}_k \vec{X}_k \quad \text{where} \quad \tilde{\alpha}_k = -\frac{\alpha_k}{\alpha_{i_*}}$$

So X_{i_*} can be written in terms of vectors from $\{X_k : k \neq i_*\}$.

4 $X_1 = \begin{pmatrix} 1 \\ 4 \\ 2 \\ -3 \end{pmatrix}$, $X_2 = \begin{pmatrix} 7 \\ 10 \\ -4 \\ -1 \end{pmatrix}$, $X_3 = \begin{pmatrix} -2 \\ 1 \\ 5 \\ -14 \end{pmatrix}$ $X_4 = \begin{pmatrix} -2 \\ 1 \\ 5 \\ -4 \end{pmatrix}$

Is $\{X_1\ X_2\ X_3\}$ independent?

$\alpha_1 X_1 + \alpha_2 X_2 + \alpha_3 X_3 = 0$

$$\begin{pmatrix} 1 & 7 & -2 \\ 4 & 10 & 1 \\ 2 & -4 & 5 \\ -3 & -1 & -14 \end{pmatrix} \begin{pmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

has augmented matrix

$$\left[\begin{array}{ccc|c} 1 & 7 & -2 & 0 \\ 4 & 10 & 1 & 0 \\ 2 & -4 & 5 & 0 \\ -3 & -1 & -14 & 0 \end{array}\right]$$

(over)

$$\sim \left[\begin{array}{ccc|c} 1 & 7 & -2 & 0 \\ 0 & -18 & 9 & 0 \\ 0 & -18 & 9 & 0 \\ 0 & 20 & -20 & 0 \end{array}\right]$$

$$\sim \left[\begin{array}{ccc|c} 1 & 7 & -2 & 0 \\ 0 & 2 & -1 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] \quad R3 = 2R3 - R2$$

$$\sim \left[\begin{array}{ccc|c} 1 & 7 & -2 & 0 \\ 0 & 2 & -1 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

Now, back substitution

$\alpha_3 = 0$

$2\alpha_2 - \alpha_3 = 0 \Rightarrow \alpha_2 = 0$

$\alpha_1 + 7\alpha_2 - 2\alpha_3 = 0 \Rightarrow \alpha_1 = 0$

so $\alpha_1 \vec{x}_1 + \alpha_2 \vec{x}_2 + \alpha_3 \vec{x}_3 = 0$

$\Rightarrow \alpha_1 = \alpha_2 = \alpha_3 = 0$

so $\{x_1\ x_2\ x_3\}$ is <u>INDEPENDENT</u>

5) Is $\{X_1\ X_2\ X_4\}$ independent?

The augmented matrix is

$$\left[\begin{array}{ccc|c} 1 & 7 & -2 & 0 \\ 4 & 10 & 1 & 0 \\ 2 & -4 & 5 & 0 \\ -3 & -1 & -4 & 0 \end{array}\right]$$

$$\sim \left[\begin{array}{ccc|c} 1 & 7 & -2 & 0 \\ 0 & -18 & 9 & 0 \\ 0 & -18 & 9 & 0 \\ 0 & 20 & -10 & 0 \end{array}\right]$$

$$\sim \left[\begin{array}{ccc|c} 1 & 7 & -2 & 0 \\ 0 & 2 & -1 & 0 \\ 0 & 2 & -1 & 0 \\ 0 & 2 & -1 & 0 \end{array}\right]$$

$$\sim \left[\begin{array}{ccc|c} 1 & 7 & -2 & 0 \\ 0 & 2 & -1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

Back substitution now

(over)

α_3 is free, say $\alpha_3 = \alpha$

$2\alpha_2 - \alpha = 0 \Rightarrow \alpha_2 = \alpha/2$

$\alpha_1 + 7\alpha_2 - 2\alpha_3 = 0$ $\qquad \frac{7-4}{2}$

$\Rightarrow \alpha_1 + 7/2\,\alpha - 2\alpha = 0$

$\alpha_1 = -3/2\,\alpha$

So take $\alpha = 2$ (for example)

to see

$$-3\vec{X}_1 + \vec{X}_2 + 2\vec{X}_4 = \vec{0}$$

and conclude $\{X_1\ X_2\ X_4\}$ is

NOT INDEPENDENT.

b) Is $\{X_1\ X_3\ X_4\}$ independent?

The augmented matrix is

$$\left[\begin{array}{ccc|c} 1 & -2 & -2 & 0 \\ 4 & 1 & 1 & 0 \\ 2 & 5 & 5 & 0 \\ 3 & -14 & -4 & 0 \end{array}\right] \qquad \text{(cont.)}$$

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$$\sim \left[\begin{array}{ccc|c} 1 & -2 & -2 & 0 \\ 0 & 9 & 9 & 0 \\ 0 & 9 & 9 & 0 \\ 0 & -20 & -10 & 0 \end{array}\right]$$

$$\sim \left[\begin{array}{ccc|c} 1 & -2 & -2 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 2 & 1 & 0 \end{array}\right]$$

$$\sim \left[\begin{array}{ccc|c} 1 & -2 & -2 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 \end{array}\right]$$

$$\sim \left[\begin{array}{ccc|c} 1 & -2 & -2 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

Back substitution

$\alpha_3 = 0$

$\alpha_2 + \alpha_3 = 0 \Rightarrow \alpha_2 = 0$

$\alpha_1 - 2\alpha_2 - 2\alpha_3 = 0 \Rightarrow \alpha_1 = 0$ (over)

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So since

$$\alpha_1 \vec{X}_1 + \alpha_2 \vec{X}_3 + \alpha_3 \vec{X}_4 = \vec{0}$$

$$\Rightarrow \alpha_1 = \alpha_2 = \alpha_3 = 0$$

conclude that

$\{X_1\ X_3\ X_4\}$ is INDEPENDENT.

7) Suppose $\{X_1 \cdots X_n\}$ is dependent

$\Rightarrow \exists\ \beta_1, \cdots \beta_n$ not all zero

such that

$$\beta_1 X_1 + \cdots + \beta_n X_n = \vec{0}$$

In fact for any $\beta \in \mathbb{R}$ this says

$$0 = \beta(\beta_1 X_1 + \cdots + \beta_n X_n)$$

$$= \beta\beta_1 X_1 + \cdots + \beta\beta_n X_n$$

and there are an infinite number of such β's.

(over)

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Since $y \in \text{span}\{X_1, \cdots X_n\}$

$\exists\ \alpha_1, \cdots \alpha_n$ such that

$$y = \alpha_1 X_1 + \cdots + \alpha_n X_n$$

But also (from above)

$$0 = \beta\beta_1 X_1 + \cdots + \beta\beta_n X_n$$

So ($\vec{y} + \vec{0} = \vec{y}$)

$$y = (\alpha_1 + \beta\beta_1) X_1 + \cdots + (\alpha_n + \beta\beta_n) X_n$$

$$\equiv \tilde{\alpha}_1 X_1 + \cdots + \tilde{\alpha}_n X_n,$$

and there are an infinite number of such $\tilde{\alpha}_1, \cdots, \tilde{\alpha}_n$.

8) $X_1 = \begin{pmatrix} 1 \\ 4 \\ 2 \\ -3 \end{pmatrix}$ $X_2 = \begin{pmatrix} 7 \\ 10 \\ -4 \\ -1 \end{pmatrix}$ $X_3 = \begin{pmatrix} -2 \\ 1 \\ 5 \\ -14 \end{pmatrix}$

a) is $Y = \begin{pmatrix} 1 \\ 2 \\ 3 \\ 4 \end{pmatrix} \in \text{span}\{X_1\ X_2\ X_3\}$?

(over)

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This asks, are there $\alpha_1\ \alpha_2\ \alpha_3$ such that

$$\alpha_1 \cdot \begin{pmatrix} 1 \\ 4 \\ 2 \\ -3 \end{pmatrix} + \alpha_2 \begin{pmatrix} 7 \\ 10 \\ -4 \\ -1 \end{pmatrix} + \alpha_3 \begin{pmatrix} -2 \\ 1 \\ 5 \\ -14 \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \\ 3 \\ 4 \end{pmatrix}$$

which is the same as

$$\begin{pmatrix} 1 & 7 & -2 \\ 4 & 10 & 1 \\ 2 & -4 & 5 \\ -3 & -1 & -14 \end{pmatrix} \begin{pmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_3 \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \\ 3 \\ 4 \end{pmatrix}$$

The augmented matrix is

$$\left[\begin{array}{ccc|c} 1 & 7 & -2 & 1 \\ 4 & 10 & 1 & 2 \\ 2 & -4 & 5 & 3 \\ -3 & -1 & -14 & 4 \end{array}\right]$$

$$\sim \left[\begin{array}{ccc|c} 1 & 7 & -2 & 1 \\ 0 & -18 & 9 & -2 \\ 0 & -18 & 9 & 1 \\ 0 & 20 & -20 & 7 \end{array}\right]$$

(over)

⑭

$$\sim \left[\begin{array}{ccc|c} 1 & 7 & -2 & 1 \\ 0 & 2 & -1 & 2/9 \\ 0 & 2 & -1 & -1/9 \\ 0 & 2 & -2 & 7/10 \end{array}\right]$$

$$\sim \left[\begin{array}{ccc|c} 1 & 7 & -2 & 1 \\ 0 & 2 & -1 & 2/9 \\ 0 & 0 & 0 & -3/9 \\ 0 & 0 & -1 & 7/10 - 2/9 \end{array}\right]$$

$$\sim \left[\begin{array}{ccc|c} 1 & 7 & -2 & 1 \\ 0 & 2 & -1 & 2/9 \\ 0 & 0 & 1 & -43/90 \\ 0 & 0 & 0 & 1/3 \end{array}\right] \qquad \frac{63}{90} - \frac{20}{90}$$

The last row says

$0\alpha_1 + 0\alpha_2 + 0\alpha_3 = 1/3$

which is impossible.

So $y \notin \text{span}\{X_1\ X_2\ X_3\}$.

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Sorry. There was a typo on y. I meant

$$y = \begin{pmatrix} 6 \\ 15 \\ 3 \\ -18 \end{pmatrix} \longleftarrow \text{woops}$$

The augmented matrix is

$$\left[\begin{array}{ccc|c} 1 & 7 & -2 & 6 \\ 4 & 10 & 1 & 15 \\ 2 & -4 & 5 & 3 \\ -3 & -1 & -14 & -18 \end{array}\right]$$

-14
-6
$-18+18$

$$\sim \left[\begin{array}{ccc|c} 1 & 7 & -2 & 6 \\ 0 & -18 & 9 & -9 \\ 0 & -18 & 9 & -9 \\ 0 & 20 & -20 & 0 \end{array}\right]$$

$$\sim \left[\begin{array}{ccc|c} 1 & 7 & -2 & 6 \\ 0 & 2 & -1 & 1 \\ 0 & 2 & -1 & 1 \\ 0 & 1 & -1 & 0 \end{array}\right]$$

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$-1+2$

$$\sim \left[\begin{array}{ccc|c} 1 & 7 & -2 & 6 \\ 0 & 2 & -1 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & -1 \end{array}\right]$$

$$\sim \left[\begin{array}{ccc|c} 1 & 7 & -2 & 6 \\ 0 & 2 & -1 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

Back substitute

$\alpha_3 = 1$

$2\alpha_2 - \alpha_3 = 1 \Rightarrow \alpha_2 = 1$

$\alpha_1 + 7\alpha_2 - 2\alpha_3 = 6 \Rightarrow \alpha_1 + 7 - 2 = 6$

$\Rightarrow \alpha_1 = 1$

So

$$y = \begin{pmatrix} 6 \\ 15 \\ 3 \\ -18 \end{pmatrix} = 1\begin{pmatrix} 1 \\ 4 \\ 2 \\ -3 \end{pmatrix} + 1\begin{pmatrix} 7 \\ 10 \\ -4 \\ -1 \end{pmatrix} + 1\begin{pmatrix} -2 \\ 1 \\ 5 \\ -14 \end{pmatrix}$$

and $y \in \text{Span}\{x_1\, x_2\, x_3\}$

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a) $x_1 = \begin{pmatrix} 1 \\ 4 \\ 2 \\ -3 \end{pmatrix}$ $x_3 = \begin{pmatrix} -2 \\ 1 \\ 5 \\ -14 \end{pmatrix}$ $x_4 = \begin{pmatrix} -2 \\ 1 \\ 5 \\ -4 \end{pmatrix}$

To get standard basis for $\text{Span}\{x_1\ x_3\ x_4\}$ consider augmented matrix

rows are x_1 x_3 x_4

$$\begin{array}{c} \begin{array}{cccc} 2 & 8 & 4 & -6 \end{array} \\ \begin{bmatrix} 1 & 4 & 2 & -3 \\ -2 & 1 & 5 & -14 \\ -2 & 1 & 5 & -4 \end{bmatrix} \end{array}$$

and reduce to row canonical form

$$\begin{bmatrix} 1 & 4 & 2 & -3 \\ 0 & 9 & 9 & -20 \\ 0 & 9 & 9 & -10 \end{bmatrix}$$

$$\sim \begin{bmatrix} 1 & 4 & 2 & -3 \\ 0 & 9 & 9 & -20 \\ 0 & 0 & 0 & 10 \end{bmatrix}$$

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$$\sim \begin{bmatrix} 1 & 4 & 2 & -3 \\ 0 & 9 & 9 & -20 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{matrix} R1 = R1 + 3R3 \\ R2 = R2 + 20R3 \\ \end{matrix}$$

$$\sim \begin{bmatrix} 1 & 4 & 2 & 0 \\ 0 & 9 & 9 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$\sim \begin{bmatrix} 1 & 4 & 2 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{matrix} R1 = R1 - 4R2 \\ \\ \end{matrix}$$

$$\sim \begin{bmatrix} 1 & 0 & -2 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

This is now in row canonical form.
Read off standard basis

$$e_1 = \begin{pmatrix} 1 \\ 0 \\ -2 \\ 0 \end{pmatrix} \quad e_2 = \begin{pmatrix} 0 \\ 1 \\ 1 \\ 0 \end{pmatrix} \quad e_3 = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}$$

(cont)

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and so

$$\text{Span}\left\{\begin{pmatrix}1\\4\\2\\-3\end{pmatrix},\begin{pmatrix}-2\\1\\0\\-14\end{pmatrix},\begin{pmatrix}-2\\1\\5\\-4\end{pmatrix}\right\}$$

$$=\text{Span}\left\{\begin{pmatrix}1\\0\\-2\\0\end{pmatrix},\begin{pmatrix}0\\1\\1\\0\end{pmatrix},\begin{pmatrix}0\\0\\0\\1\end{pmatrix}\right\}$$

BTW, notice. This is a check

$$X_1=\begin{pmatrix}1\\4\\2\\-3\end{pmatrix}=e_1+4e_2-3e_3=\begin{pmatrix}1\\0\\-2\\0\end{pmatrix}+\begin{pmatrix}0\\4\\4\\0\end{pmatrix}+\begin{pmatrix}0\\0\\0\\-3\end{pmatrix}$$

$$X_2=\begin{pmatrix}-2\\1\\5\\-14\end{pmatrix}=-2e_1+e_2-14e_3=\begin{pmatrix}-2\\0\\+4\\0\end{pmatrix}+\begin{pmatrix}0\\1\\1\\0\end{pmatrix}+\begin{pmatrix}0\\0\\0\\-14\end{pmatrix}$$

$$X_3=\begin{pmatrix}-2\\1\\5\\-4\end{pmatrix}=-2e_1+e_2-4e_3=\begin{pmatrix}-2\\0\\4\\0\end{pmatrix}+\begin{pmatrix}0\\1\\1\\0\end{pmatrix}+\begin{pmatrix}0\\0\\0\\-4\end{pmatrix}$$

10) $X1=\begin{pmatrix}1\\4\\-2\\-3\end{pmatrix}\quad X_2=\begin{pmatrix}7\\16\\-4\\-1\end{pmatrix}\quad X_4=\begin{pmatrix}-2\\1\\5\\-4\end{pmatrix}$

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The augmented matrix to consider is

$$\begin{bmatrix} 1 & 4 & 2 & -3 \\ 7 & 10 & -4 & -1 \\ -2 & 1 & 5 & -4 \end{bmatrix} \quad 2 \quad 8 \quad 4 \quad -6$$

$$\sim \begin{bmatrix} 1 & 4 & 2 & -3 \\ 0 & -18 & -18 & 20 \\ 0 & 9 & 9 & -10 \end{bmatrix}$$

$$\sim \begin{bmatrix} 1 & 4 & 2 & -3 \\ 0 & 9 & 9 & -10 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Pivot

$$\sim \begin{bmatrix} 1 & 4 & 2 & -3 \\ 0 & 1 & 1 & -10/9 \\ 0 & 0 & 0 & 0 \end{bmatrix} \quad R1 = R1 - 4R2$$

$$\sim \begin{bmatrix} 1 & 0 & -2 & 13/9 \\ 0 & 1 & 1 & -10/9 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

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disregard the 0 row and set

$$e_1 = \begin{pmatrix} 1 \\ 0 \\ -2 \\ 13/9 \end{pmatrix} \text{ and } e_2 = \begin{pmatrix} 0 \\ 1 \\ 1 \\ -10/9 \end{pmatrix}$$

so

$$\boxed{\text{Span}\left\{ \begin{pmatrix} 1 \\ 4 \\ 2 \\ -3 \end{pmatrix}, \begin{pmatrix} 7 \\ 10 \\ -4 \\ -1 \end{pmatrix}, \begin{pmatrix} -2 \\ 1 \\ 5 \\ -4 \end{pmatrix} \right\} = \text{Span}\left\{ \begin{pmatrix} 1 \\ 0 \\ -2 \\ 13/9 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 1 \\ -10/9 \end{pmatrix} \right\}}$$

Check this by observing

$$x_1 = \begin{pmatrix} 1 \\ 4 \\ 2 \\ -3 \end{pmatrix} = e_1 + 4e_2 = \begin{pmatrix} 1 \\ 0 \\ -2 \\ 13/9 \end{pmatrix} + \begin{pmatrix} 0 \\ 4 \\ 4 \\ -40/9 \end{pmatrix}$$

$$x_2 = \begin{pmatrix} 7 \\ 10 \\ -4 \\ -1 \end{pmatrix} = 7e_1 + 10e_2 = \begin{pmatrix} 7 \\ 0 \\ -14 \\ \frac{13 \cdot 7}{9} \end{pmatrix} + \begin{pmatrix} 0 \\ 10 \\ 10 \\ -\frac{100}{9} \end{pmatrix}$$

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$$x_3 = \begin{pmatrix} -2 \\ \frac{1}{5} \\ -4 \end{pmatrix} = -2e_1 + e_2 = \begin{pmatrix} -2 \\ 0 \\ 4 \\ -\frac{26}{9} \end{pmatrix} + \begin{pmatrix} 0 \\ 1 \\ 1 \\ -\frac{10}{9} \end{pmatrix}$$

So my answer checks out.